Unit-I

(Module- I)

Integral transform, Z- transform.

- Fourier Integral fourier transform
- Complex fourier transform
- Convolution theorem (without proof)
- Fourier sine and cosine transformation
- Application of fourier transform to simple one dimensional heat equation, wave equation and Laplace equation

  Z- transform and its application to solve diffean equation

Chapter-1

Integral Transforms

Integral transforms are used in the solution of partial differential equation. The choice of a particular transform to be employed for the solution of equation depends on the boundary conditions of the problem.

A fourier transform when applied to a partial differential equation reduces the number of its independent variable by one.

DEFINITION-

The integral transform of a function$f(x)$ denoted by $I\{f(x)\}$ is defined as

$$I\{F(x)\} - f(p) = \int_a^b F(x).k(p,x)dx$$

Where $k(p,x)$ is a known function of p and x called the **Kemel** of the transform p is called the **parameter** of the transform and $F(x)$

is called **inverse transform** of $f(p)$.
Some of well-known transform are given below.

(i) **Laplace transform-** let $k(p,x) = e^{-px}$

$$L\{F(x)\} = f(p) = \int_0^\infty F(x)e^{-px}dx$$

(ii) **Fourier Complex transform –** let $k(p,x) = e^{ipx}$

$$F\{F(x)\} = f(p) = \int_{-\infty}^\infty F(x)e^{ipx}dx$$

(iii) **Fourier sine transform-** let $k(p,x) = \sin px$

$$F_s\{f((x)\} = f(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinpx\,dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(p).sinpx\,dp$$ **(Inversion formula)**

(iv) **Fourier cosine transform-** let $k(p,x) = \cos px$

$$F_c\{f((x)\} = f(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cospxdx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(p).cospx\,dp$$ **(Inversion formula)**

(v) **Hankel Transform-** let $k(p,x) = xJ_n(px)$

$$\boldsymbol{H_n\{f((x)\} = f(p) = \int_0^{\infty} f(x).\, xJ_n(px)dx}$$

Where $J_n(px)$ in the Bessel function of the First kind of order n.

(vi) **Mellin Transform-** let $k(p,x) = x^{p-1}$

$$\boldsymbol{M\{f((x)\} = f(p) = \int_0^{\infty} f(x).\, x^{p-1} dx}$$

(vii) **Hilbert Transform-** let $k(p,x) = \frac{1}{p-x}$

$$\boldsymbol{H\{f((x)\} = f(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{p-x} dx}$$

FOURIRER INTEGRAL THEOREM

Statement If $F(x)$ satisfies Dirichlets condition in every interval $(-c,c)$ and $\int_{-\infty}^{\infty}|f(x)|dx$ be converges then

$$F(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} F(t)\ cos\lambda(t-x)dtd\lambda$$

Proof : Let a function $f(x)$ which satisfies the Dirichlets condition in every integral $(-c,c)$ so that

$F(x) = \frac{a_0}{z} + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{c} + \mathrm{b_n sin}\frac{n\pi x}{c})$ -------------------(i)

Where,

$$a_0 = \frac{1}{c}\int_{-c}^{c} F(t)dt$$

$$a_n = \frac{1}{c}\int_{-c}^{c} F(t)cos\frac{n\pi t}{c}dt$$

$$b_n = \frac{1}{c}\int_{-c}^{c} F(t)sin\frac{n\pi t}{c}dt$$

Substituting the the value of a_o, a_n and b_n in equation (i) we get

$$F(x) = \frac{1}{2c}\int_{-c}^{c} F(t)dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c}\left[cos\frac{n\pi x}{c}.\, cos\frac{n\pi t}{c} + sin\frac{n\pi x}{c}.\, sin\frac{n\pi t}{c}\right]F(t)dt$$

$= \frac{1}{2c}\int_{-c}^{c} F(t)dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c}\left[cos\,\frac{n\pi(t-x)}{c}.\,F(t)dt\right]$ (using formula $\cos(A-B)$)

$= \frac{1}{2c}\int_{-c}^{c} F(t)\left\{1 + 2\sum_{n=1}^{\infty}\int_{-c}^{c} cos\,\frac{n\pi(t-x)}{c}\,dt\right\}$ -------------(2)

$sin\theta\ cosine$ function are even function ie. $\cos(-\theta) = +cos\theta$ the expresson form equation (2)

$$1 + 2\sum_{n=1}^{\infty} cos\,\frac{n\pi(t-x)}{c} = \sum_{n=-\infty}^{\infty} cos\,\frac{n\pi(t-x)}{c}$$

Therefore (2) becomes

$$F(x) = \frac{1}{2c}\int_{-c}^{c} F(t)\left\{\sum_{n=-\infty}^{\infty} cos\,\frac{n\pi(t-x)}{c}\right\}dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} F(t)\left\{\frac{\pi}{c}\sum_{-\infty}^{\infty} cos\,\frac{n\pi(t-x)}{c}\right\}dt \text{ ---------------(3)}$$

Let us suppose that c increase indefinitely, so we can write $\frac{n\pi}{c} = \lambda$ and $\frac{\pi}{c} = d\lambda$ then assumption gives-

$$Lim_{c\to 0}\left\{\frac{\pi}{c}\sum_{-\infty}^{\infty} cos\,\frac{n\pi}{c}(t-x)\right\} = \int_{-\infty}^{\infty} cos\ \lambda(t-x)d\lambda$$

$$= 2\int_{0}^{\infty} cos\ \lambda(t-x)d\lambda \text{-------------(4)}$$

Substiting equation (4) in equation (3)

We get

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\left\{2\int_{0}^{\infty} cos\,\lambda(t-x)d\lambda\right\}dt$$

$F(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} F(t)\,cos\,\lambda(t-x)d\lambda dt$ -----------(5)

Which is known as Fourier integral of $F(x)$ equation (5) is five at a point of continuity

FOURIRER SINE AND COSINE INTEGRAL THEOREM

We known that

$$cos\ \lambda(t-x) = cos\ (\lambda t - \lambda x)$$

$$= cos\lambda t\ cos\lambda x + sin\lambda t\ sin\lambda x$$

Then Fourier integral can be written as

$$F(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty F(t)\cos\lambda(t-x)dtd\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty F(t)\{cos\lambda t\ cos\lambda x + sin\lambda t\ sin\lambda x\}dtd\lambda$$

$$= \frac{1}{\pi}\int_0^\infty cosdx \int_{-\infty}^\infty F(t)cos\lambda t\ d\lambda\ dt + \frac{1}{\pi}\int_0^\infty sindx \int_{-\infty}^\infty F(t)sin\lambda t\ d\lambda\ dt$$

---------------(1)

Case I When $\boldsymbol{F(x)}$is an odd function

$\boldsymbol{F(t)cos}\ \lambda t$ is odd while $F(t)sin\lambda t$ is even thus the first integral in (1) varishes

$$\left\{\begin{matrix} \boldsymbol{for\ odd\ function} & \int_{-a}^{a} \boldsymbol{f(x)dx = 0} \\ \boldsymbol{for\ even\ fuction} & \int_{-a}^{a} \boldsymbol{f(x)dx = 2\int_0^a f(x)dx} \end{matrix}\right\}$$

We get $\boldsymbol{F(x)} = \frac{2}{\pi}\int_0^\infty sin\lambda x \int_0^\infty F(t)sin\lambda t\ d\lambda\ dt$

This is called **Fourier sine integral.**

Case II When $\boldsymbol{F(x)}$ is an even function

Since $\boldsymbol{F(t)cos}\ \lambda t$ is even while $F(t)sin\lambda t$ is odd thus the second integral in (1) varishes

we get $\boldsymbol{F(x)} = \frac{2}{\pi}\int_0^\infty cos\lambda x \int_0^\infty F(t)cos\lambda t\ d\lambda\ dt$

This is called **Fourier cosine integral**

FOURIRER COMPLEX INTEGRAL

Fourier integral $f(x)$ is

$$F(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty F(t)\cos\lambda(t-x)dtd\lambda \text{-------------(1)}$$

$sin\theta\ cos\lambda(t-x)$ is an even function of λ, we have from (1)

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t)\, cos\, \lambda(t-x) dt d\lambda \text{-------------}(2)$$

Also $sin\theta\ sin\lambda(t-x)$ is an odd function λ

We have

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t)\, sin\, \lambda(t-x) dt d\lambda \text{-------------}(3)$$

$$\left\{\begin{matrix} \boldsymbol{for\ odd\ function} & \boldsymbol{\int_{-a}^{a} f(x)dx = 0} \\ \boldsymbol{for\ even\ fuction} & \boldsymbol{\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx} \end{matrix}\right\}$$

Multiplying (3) by I and adding with (2) we get

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t)\,\{\cos\lambda(t-x) + \sin\lambda(t-x)\}\ dt d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t)\, e^{i\lambda(t-x)} \mathrm{dt\ d}\,\lambda$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda x}\, d\lambda \int_{-\infty}^{\infty} F(t) e^{i\lambda t}\, dt$$

This is known as Fourier complex integral

Example (1) Express $f(x) = 1 \quad for\ 0 \leq x \leq \pi$

$$= 0\ for\ x\ > \pi$$

As a fourier sine integral and hance evatuat

$$\int_{0}^{\infty} \frac{1-\cos(\pi-\lambda)}{\lambda}\ \sin(x\lambda) d\lambda$$

Solution :-

The fourier Sine integral for

$$\boldsymbol{F(x)} = \frac{2}{\pi}\int_{0}^{\infty} sin\ \lambda x\ d\lambda \int_{0}^{\infty} F(t) sin\lambda t\ d\lambda\ dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} sin\ \lambda x\ d\lambda \int_{0}^{\pi} sin\lambda t\ dt$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin(\lambda x) d\lambda \left[\frac{-cos\lambda t}{\lambda}\right]_0^{\pi}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos(\lambda x)}{\lambda} \sin(\lambda x) d\lambda$$

Hence $\int_0^{\infty} \frac{1-\cos(\lambda x)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2}\ f(x)$

$$= \begin{cases} \frac{\pi}{2} & for\ 0 \le x < \pi \\ 0 & for\ x > \pi \end{cases}$$

At $x = \pi$ which in point of dis continuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi - 0) + f(\pi + 0)}{2}\right]$$

$$\int_0^{\infty} \frac{1 - cos\lambda\pi}{\lambda}\ sin\lambda x\ d\lambda = \frac{\pi}{2}\left[\frac{1+0}{2}\right] = \frac{\pi}{4}$$

Example (2) Express the function

$$f(x) = \begin{cases} 1 & when\ |x| \le 1 \\ 0 & when\ |x| > 1 \end{cases}$$

As a fourier integral Hence evaluate

$$\int_0^{\infty} \frac{sin\lambda\ cos\lambda x}{\lambda}\ d\lambda$$

Solution The fourier integral for $f(x)$ is

$$F(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t) \cos\lambda(t-x) dt d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1} \cos\lambda(t-x) dt d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\frac{sin\lambda(t-x)}{\lambda}\right]_{-1}^{1} d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{[sin\lambda(1-x) - sin\lambda(-1-x)]}{\lambda}\ d\lambda$$

$$F(x) = \frac{1}{\pi}\int_0^{\infty} \frac{[sin\lambda(1+x) + sin\lambda(1-x)]}{\lambda}\ d\lambda$$

$$F(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\lambda\ cos\lambda x}{\lambda}\ d\lambda$$

Hence $\int_0^{\infty} \frac{\sin\lambda\ cos\lambda x}{\lambda}\ d\lambda = \frac{\pi}{2}\ F(x)$

$$= \begin{cases} \frac{\pi}{2} & for\ |x| < 1 \\ 0 & for\ |x| > 1 \end{cases}$$

At $|x| = 1$ i.e $x \pm 1$

$F(x)$ is discontinuous and integral has the value $\frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}$

Note, Putting $x = 0$ we get

$$\int_0^{\infty} \frac{\sin\lambda}{\lambda}\ d\lambda = \frac{\pi}{2}$$

or

$$\int_0^{\infty} \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

Example (3) Using fourier integral theroum to prove that

(i) Fourier fine transform of $e^{-|x|}$ show that $\int_0^{\infty} \frac{cosx\omega}{1+\omega^2}\ d\omega = \frac{\pi}{2}\ e^{-x} \quad (x > 0)$

(ii) Frurier cosine integral of $e^{-|x|}$ to show that $\int_0^{\infty} \frac{cosx\omega}{1+\omega^2}\ d\omega = \frac{\pi}{2}\ e^{-x} \quad (x \geq 0)$

Proof In the interval $(0, \infty)$, x is always positive therefore $e^{-|x|} = e^{-x}$

(i) Fourier Sine Integral is

$$F(x) = \frac{2}{\pi}\int_0^{\infty} sin\lambda x \int_0^{\infty} F(t) sin\lambda t dt\ d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} sin\lambda x\ d\lambda \int_0^{\infty} e^{-t} sin\lambda t\ dt$$

$$= \frac{2}{\pi}\int_0^{\infty} sin\lambda x\ d\lambda \left[\frac{e^{-t}}{1+x^2}(-sin\lambda t - \lambda cos\lambda t)\right]_0^{\infty}$$

$$= \frac{2}{\pi}\int_0^\infty sin\lambda x \left(\frac{\lambda}{1+\lambda^2}\right) d\lambda$$

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda sin\lambda x}{1+\lambda^2}\ d\lambda$$

There fore $\int_0^\infty \frac{\lambda sin\lambda x}{1+\lambda^2}\ d\lambda = \frac{\pi}{2}e^{-x}\ , x > 0$

or $\int_0^\infty \frac{\omega sin\omega x}{1+\omega^2}\ d\omega = \frac{\pi}{2}e^{-x}\ , x > 0$

(ii) Fourier cosine integral is

$$F(x) = \frac{2}{\pi}\int_0^\infty cos\lambda x \int_0^\infty F(t) cos\lambda t\ dt\ d\lambda$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty cos\lambda x \int_0^\infty e^{-t} cos\lambda t\ dt\ d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty cos\lambda x\ d\lambda \left[\frac{e^{-t}}{1+\lambda^2}(-cos\lambda t + \lambda sin\ \lambda t)\right]_0^\infty$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty cos\lambda x\ d\lambda \left(\frac{1}{1+\lambda^2}\right)$$

there fore $\int_0^\infty \frac{cos\lambda x}{1+\lambda^2}\ d\lambda = \frac{\pi}{2}e^{-x}\ , x \geq 0$

or $\int_0^\infty \frac{cos\omega x}{1+\omega^2}\ d\omega = \frac{\pi}{2}e^{-x}\ , x \geq 0$

Example (4) Using fourier integral representation show that

$$\int_0^\infty \frac{cosx\omega + \omega sin\omega x}{1+\omega^2}\ d\omega = \begin{cases} 0 & if\ x < 0 \\ \frac{\pi}{2} & if\ x = 0 \\ \pi e^{-x} & if\ \ x > 0 \end{cases}$$

Solution We have

$\int_0^\infty \frac{cosx\omega}{1+\omega^2}\ d\omega = \frac{\pi}{2}\ e^{-x}\quad (x > 0)$-------------(1)

and $\int_0^\infty \frac{\omega sin\ x\omega}{1+\omega^2}\ d\omega = \frac{\pi}{2}\ e^{-x}\quad (x > 0)$----------(2)

Adding eq (1) and (2) we get

$$\int_0^\infty \frac{cosx\omega + \omega sin\omega x}{1+\omega^2}\ d\omega = \frac{\pi}{2}\ e^{-x} + \frac{\pi}{2}\ e^{-x}$$

$$= \left(\frac{\pi}{2} + \frac{\pi}{2}\right) e^{-x}, x > 0$$

$$= \pi e^{-x}, x > 0$$

When $x = 0$ $\int_0^\infty \frac{cosx\omega+\omega sin\omega x}{1+\omega^2}\ d\omega = \int_0^\infty \frac{d\omega}{1+\omega^2}$

$$= [\tan^{-1}\omega]_0^\infty = \frac{\pi}{2}$$

When $x < 0$ $\int_0^\infty \frac{cosx\omega+\omega sin\omega x}{1+\omega^2}\ d\omega = \int_0^\infty \frac{cosx\omega-\omega sin\omega x}{1+\omega^2}\ d\omega$

$$= \frac{\pi}{2} e^{-x} - \frac{\pi}{2} e^{-x} = 0$$

Hence $$\int_0^\infty \frac{cosx\omega+\omega sin\omega x}{1+\omega^2}\ d\omega = \begin{cases} 0 & if\ x < 0 \\ \frac{\pi}{2} & if\ x = 0 \\ \pi e^{-x} & if\ \ x > 0 \end{cases}$$

Example (5) find the complex form of the fourier integral represtntation of

$$f(x) = \begin{cases} e^{-kx} & x > 0\ and\ k > 0 \\ 0 & otherwise \end{cases}$$

Solution we have

The complex form of fourier integral –

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} \int_{-\infty}^{\infty} f(t) e^{i\lambda t}\ \ dt\ d\lambda$$ -------------------(1)

Here given

$$f(t) = \begin{cases} e^{-kt} & t > 0\ and\ k > 0 \\ 0 & otherwise \end{cases}$$

From (1)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} \left[\int_0^\infty e^{-kt} e^{i\lambda t}\ \ dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} \left[\int_0^\infty e^{-(k-i\lambda)t}\ \ dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\, d\lambda \left[\frac{e^{-(k-i\lambda)t}}{-(k-i\lambda)}\right]_0^{\infty}$$

$$= \frac{1}{2\pi}\int e^{-i\lambda x}\, d\lambda \left\{\frac{-1}{k-i\lambda}\right\}[e^{-\infty} - e^{0}]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{k-i\lambda}\, d\lambda$$

Exercise ( )

1. Find fourier sine integral for $f(x) = e^{-ax}$ **Ans** $\boldsymbol{\frac{2}{\pi}\int_0^{\infty} \frac{\lambda\, sin\lambda x}{a^2+\lambda^2}\, d\lambda}$

2. Using fourier integral show that

$$\int_0^{\infty} \frac{\lambda\, sin\lambda x}{\lambda^2 + k^2}\, d\lambda = \frac{\pi}{2} e^{-kx} \qquad (x > 0\,, k > 0)$$

3. Find the fourier cosine integral of the function e^{-ax} hence show that

$$\int_0^{\infty} \frac{cos2x}{\lambda^2 + 1^2}\, d\lambda = \frac{\pi}{2} e^{-x} \qquad x \geq 0$$

Ans $\frac{2a}{\pi}\int_0^{\infty} \frac{cos\lambda x}{\lambda^2+a^2} d\lambda$

4. Find the fourier sine transform of $e^{-|x|}$ Hence evaluate

$$\int_0^{\infty} \frac{xsinmx}{1 + x^2}\, dx$$

Ans $\boldsymbol{\frac{p}{1+p^2}\frac{\pi}{2}e^{-m}}$

5. Show that the fourier sine transform of $\frac{x}{1+x^2}$ is $\sqrt{\frac{\pi}{2}}$ as e^{-as}

6. Show that the fourier sine transform of $\frac{1}{1+x^2}$ is $\sqrt{\frac{\pi}{2}}$ as e^{-p}

7. Find the sine cosine transform ofe^{-ap} $(a > 0)$

Ans $\frac{p}{a^2+p^2}$, $\frac{a}{a^2+p^2}$

8. Find the fourier sine and cosine transform of $(ae^{-ax} + be^{\beta x})$

Ans $\left(\frac{ap}{p^2+\propto^2}, \frac{bp}{p^2+\beta^2}\right)\left(\frac{a\propto}{p^2+\propto^2}, \frac{b\beta}{p^2+\beta^2}\right)$

9. Using fourier integral thermo provethat

$$\int_0^{\infty} \left(\frac{\lambda^2 + 2}{\lambda^4 + 4}\right) cos\lambda x d\lambda = \frac{\pi}{2} e^{-x} cosx \; if \; x > 0$$

10. Using fourier integral method prove that

$$\int_0^\infty \left(\frac{\sin\pi\lambda}{1-\lambda^2}\right)\sin\lambda x d\lambda = \begin{cases} \frac{\pi}{2}\sin x & if\ \ 0 \le x \le \pi \\ 0 & if x > \pi \end{cases}$$

Complex Fourier Transform

TheComplex form of fourier integral $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-px}dp \int_{-\infty}^{\infty} f(t)e^{ipt}dt$ It follows that is

$f(p) = \int_{-\infty}^{\infty} f(t)e^{-ipt}dt$------------(1)

in depind as the fourier transform of F(t) and is denoted by $F(p)$. The function $f(t)$ is called the inverse fourier transform of $f(p)$

The inverse formula for fourier transform give

$$F^{-1}\{f(p)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipt}dt$$

For reason of summery , we multiply both $F(t)$ and $F(p)$ by $\sqrt{\frac{1}{2\pi}}$ instead of having the factor $\frac{1}{2\pi}$ in only one function. Thus we obtain the definition of fourier transform as

$$f(p) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ipt}dt$$

$$f(t) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(p)e^{-ipt}dt$$

Fourier Sine and Cosine Transform

The infinite fourier transform of the function$F(x)$in $0 < x < \infty$

The fourier sine integral

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} sinpx\, dx \int_0^{\infty} f(t) sinpt\, dt$$

The fouier cosine integral

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} cospx\, dx \int_0^{\infty} f(t) cospt\, dt$$

The fourier sine transform of the function is denoted by

$$F_s(p) = \int_0^{\infty} f(x) sinpx\, dx$$

And the function $F(x)$ is called the inverse fouier sine transform of $F_s\{f(x)\}$ The inverse formula is

$$F(x) = F^{-1}\{f_s(p)\} = \frac{2}{\pi}\int_0^{\infty} f_s(p) sinpx\, dp$$

The fourier cosine transform of the function in denoted by $F_c\{f(x)\}$ or $F_c(p)$ as-

$$F_c(p) = F_c\{f(x)\} = \int_0^{\infty} F(x) cos\, px\, dx$$

And the function $F(x)$ is called inverse fourier cosine transform of $F_c(p)$

The inverse formula for infinite fourier cosine transform in given by

$$F(x) = F^{-1}\{f_c(p)\} = \frac{2}{\pi}\int_0^{\infty} f_c(p) cospx\, dp$$

Properties Of Fourier Transforms

(1)Linear Property

Statemant:- If $F(s)$ and $G(s)$ are fourier transform of $f(x)$ and $g(x)$ repectively then

$$F[af\{x\} + dg(x)] = aF(s) + bG(s)$$

Where a and b are constant

Proof- We have

$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$---------(1)

and

$$G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)dx \text{---------(2)}$$

$$F[af(x) + bg(x)]$$

$$= \int_{-\infty}^{\infty} e^{isx} \ [af(x) + bg(x)]dx$$

$$= a\int_{-\infty}^{\infty} e^{isx} f(x)dx + b\int_{-\infty}^{\infty} e^{isx} g(x)dx$$

$$= aF(s) + bG(s) \qquad \text{using (1) and (2)}$$

Change of Scale Property

Statemant:- If $F(s)$ is the complex fourier transform of $f(x)$. then

$$F\left[af\{x\} = \frac{1}{a}F(\frac{s}{a})\right], \ \ a \neq 0$$

Proof- We know that

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx \text{---------(1)}$$

$$F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax)dx$$

$$= \int_{-\infty}^{\infty} e^{i(st/a)} f(t)\frac{dt}{a}$$

$$=\frac{1}{a}\int_{-\infty}^{\infty} e^{i(st/a)} f(t)dt$$

$$= \frac{1}{a} \ F \ (s/a)$$

Note:- If $F_s(s)$ and $F_c(s)$ are the fourier sine and cosine transform of $f(x)$ respectively then

$$F_s\{f(ax)\} = \frac{1}{a}F_s(s/a)$$

and
$$F_c\{f(ax)\} = \frac{1}{a}F_c(s/a)$$

(1) Shifting Property

Statement:- If $F(s)$ is the cmplex fourier transform of $f(x)$ then

$$F\{f(x-a)\} = e^{isa}F(s)$$

Proof: We know that

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx \text{---------(1)}$$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)dx$$

$$= \int_{-\infty}^{\infty} e^{is(t-a)} f(t)dt$$

$$= e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t)dt$$

$$F\{f(x-a)\} = e^{isa}F(s) \quad \text{using (1)}$$

Similarly

$$F(s+a) = F\{e^{iax}f(x)\}$$

(4) Modulation Theorem

Statement:- If $F(s)$ in the complex fourier transform of $f(x)$ then

$$F\{f(x)cosax\} = \frac{1}{2}[F(s+a) + F(s-a)]$$

Proof: We have

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$$

$$\therefore F\{f(x)cosax\} = \int_{-\infty}^{\infty} e^{isx} f(x)cosaxdx$$

$$= \int_{-\infty}^{\infty} e^{isx} f(x)\left(\frac{e^{iax} + e^{-iax}}{2}\right)dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)dx\right]$$

$$= \frac{1}{2}[F(s+a) + F(s-a)]$$

This result has application in radio and television where the harmonic carrier wave is modulated by an envelope

Note

If $F_s(s)$ and $F_c(s)$ are fourier sine and cosine transform of $f(x)$ respectively, then

(i) $F_s\{f(x)cosax\} = \frac{1}{2}\ [F_s(s+a) + F_s(s-a)]$

(ii) $F_c\{f(x)sinax\} = \frac{1}{2}\ [F_s(s+a) - F_s(s-a)]$

(iii) $F_s\{f(x)sinax\} = \frac{1}{2}\ [F_c(s-a) - F_s(s+a)]$

(2) If $F\{f(x)\} = F(s)$ then

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s)$$

Proof We know that $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx - - - - - - - - (1)$

Differentiating eq (1) w. r to s both side ritmes we get

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n e^{isx} f(x) dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x)^n e^{isx} f(x) dx$$

$$=(i)^n F x^n f(x)$$

$$F x^n f(x) = (-i)^n \frac{d^n F(s)}{ds^n}$$

Some Important Result

(1) $\int_0^\infty \frac{\sin mx}{x}\ dx = \frac{\pi}{2}\ , (m > 0)$

(2) $\int_0^\infty\ e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

(3) $\int_0^\infty\ \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} dx = \frac{1}{2} \sec \frac{a}{2}\ \ (-\pi < a < \pi)$

(4) $\int_0^\infty\ \frac{e^{ax} - e^{-iax}}{e^{\pi x} - e^{-\pi x}} dx = \frac{1}{2} \tan \frac{a}{2}\ \ (-\pi < a < \pi)$

CONVOLUTION

The Convolution of two function $F(x)$ and $G(x)$ over the interval $(-\infty, \infty)$ is defined as

$$F * G = \int_{-\infty}^{\infty} F(\mu) G(x-\mu) d\mu$$

Convolution theorem for transfoms

Statemant: The fourier transform of then convolution of $f(x)$and $g(x)$ is the product of their Fourier transform i.e

$$F[f(x) * g(x)] = F[f(x)].F[g(x)]$$

Proof: We know that

$$F[f(x) * g(x)] = \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} F(\mu) g(x-\mu) d\mu \text{----------(1)}$$

Taking fourier transform of both side of (1) we have

$$F[f(x) * g(x)] = F\left[\int_{-\infty}^{\infty} F(\mu) g(x-\mu) d\mu\right]$$

$$= \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2}\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu) g(x-\mu) d\mu\right] e^{isx} dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} F(\mu) d\mu \frac{1}{\sqrt{2}\pi}\left[\int_{-\infty}^{\infty} g(x-\mu) e^{isx} dx\right]$$

$$= \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} F(\mu) d\mu \,.\, F g(x-\mu)$$

$$= \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} F(\mu) d\mu \,.\, e^{i\mu x} G(s)$$

$$G(S).\frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} F(\mu)\,.\, e^{i\mu x} d\mu$$

$$= G(s).F(s)$$

$$F[f(x)g(x)] = G(s)F(s) \quad \text{Proved by invrsion}$$

$$F^{-1}\{F(s)G(s)\} = f * g$$

$$= F^{-1}\{F(s)\} * F^{-1}\{G(s)\}$$

PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the fourier transform of $f(x)$ and $g(x)$ be $F(s)$ and $G(s)$ respectively then

(i) $\int_{-\infty}^{\infty} F(s)\bar{G}(s)ds = \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx$
Where $\bar{G}(s)$ is the complex conjucate of $G(s)$ and $\bar{g}(x)$ is the complex conjucate of $g(x)$

(ii) $\int_{-\infty}^{\infty}[F(s)]^2 ds = \int_{-\infty}^{\infty}|F(x)|^2 dx$

Proof: (i)

$$\int_{-\infty}^{\infty} F(s)\bar{G}(s)ds = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{G}(s)e^{isx}ds\right]ds$$

Since $\bar{g}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{G}(s)e^{isx}ds$

$$\int_{-\infty}^{\infty} f(x)\bar{g}(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{G}(s)\,ds.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Since $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx = F(s)$ Fourier transform

$$= \int_{-\infty}^{\infty} F(s)\bar{G}(s)ds \text{--------------}(1)$$

Putting $g(x) = f(x)$ in (1) we get

(iii) $\int_{-\infty}^{\infty} F(s)\bar{F}(s)ds = \int_{-\infty}^{\infty} f(x)\bar{f}(x)dx$
$\int_{-\infty}^{\infty}[F(s)]^2 ds = \int_{-\infty}^{\infty}[f(x)]^2 dx$

PARSEVAL'S IDENTITY FOR COSINE TRANSFORMS

(i) $\frac{2}{\pi}\int_0^{\infty} F_c(s)G_c(s)ds = \int_0^{\infty} f(x).g(x)dx$

(ii) $\frac{2}{\pi}\int_0^{\infty}|F_c(s)|^2 dx = \int_0^{\infty}|f(x)|^2 dx$

$$\therefore \int_0^{\infty}\frac{1-cos\lambda\pi}{\lambda}sin\lambda x\,d\lambda = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2} & for\ 0 \le x \le \pi \\ 0 & for\quad x > \pi \end{cases}$$

At $x = \pi$ Which is a point of discontinuity of $f(x)$ the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi - 0) + f(\pi + 0)}{2}\right] = \frac{\pi}{2}\left(\frac{1+0}{2}\right) = \frac{\pi}{4}$$

Example (2) find the fourier transform of following function

(i) $F(x) = \begin{cases} e^{i\omega x} & for\ a < x < b \\ 0 & otherwise \end{cases}$

(ii) $F(x) = e^{-|x|}$

(iii) $F(x) = e^{-a|x|} \quad , a > 0$

(iv) $f(x) = \begin{cases} x & for\ \ |x| < a \\ 0 & for\ \ |x| > a \end{cases}$

Solution: (i) we have fourier complex transform

$$f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx$$

$$= \int_{a}^{b} e^{i\omega x}e^{ipx}\,dx = \int_{a}^{b} e^{i(\omega x + px)}dx$$

$$= \left[\frac{e^{i(\omega x + px)}}{i(\omega + p)}\right]_{a}^{b}$$

$$= \frac{e^{i(\omega+p)b} - e^{i(\omega+p)a}}{i(\omega + p)}$$

(iii) We have fourier complex transform

$$f(b) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx$$

$$= \int_{-\infty}^{\infty} e^{-|x|}e^{ipx}dx$$

$$= \int_{-\infty}^{0} e^{x}e^{ipx}dx + \int_{0}^{\infty} e^{-x}e^{ipx}dx$$

$$= \int_{-\infty}^{0} e^{(1+ip)x} dx + \int_{0}^{\infty} e^{(-1+i)px} dx$$

$$= \left[\frac{e^{(1+ip)x}}{1+ip}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-ip)x}}{-(1-ip)}\right]_{0}^{\infty}$$

$$= \frac{1}{1+ip} + \frac{1}{1-ip} = \frac{2}{1+p^2}$$

(iv) In above we have just proved that

$$F\left(e^{-|x|}\right) = \left(\frac{2}{1+(p)^2}\right)$$

Using change of scale property we get-

$$F\left(e^{-a|x|}\right) = \frac{1}{a}\left(\frac{2}{1+(b/a)^2}\right) = \frac{2a}{p^2+a^2}$$

(v) $f(x) = \begin{cases} x \; for \;\; |x| < a \\ 0 \;\; for \; |x| > a \end{cases}$

We know that

Fourier complex ransform

$$f(p) = \int_{-\infty}^{\infty} F(x) e^{ipx} dx$$

$$= \int_{-a}^{a} x. e^{ipx} dx$$

$$= \int_{-a}^{a} x. (cospx + sinpx) dx$$

$$= 2i \int_{0}^{a} xsinpx \; dx$$

$$= 2i\left[\left\{x\left(\frac{-cospx}{p}\right)\right\}_{0}^{a} - \int_{0}^{a} 1. \left(\frac{-cospx}{p}\right) dx\right]$$

$$= x\left[\frac{-a}{p} cospx + \frac{1}{p}\left(\frac{sinpx}{p}\right)_{0}^{a}\right]$$

$$= x\left[\frac{-a}{p} cospx + \frac{1}{p^2} sinpx\right]$$

$$= \frac{x}{p^2}(sinpx - apcosap)$$

Example (3) find the fourier transform

$$(x) = \begin{cases} 1 & for \quad |x| < a \\ 0 & for\ |x| > a \end{cases}$$

Hence evaluate

(i) $\int_{-\infty}^{\infty} \frac{sinapcospx}{p}\, dp$ (ii) $\int_{0}^{\infty} \frac{sinp}{p}\, dp$

Solution: we know that fourier complex transform

$$f(p) = \int_{-\infty}^{\infty} F(x) e^{ipx} dx$$

$$= \int_{-a}^{a} 1.\, e^{ipx} dx = \int_{-a}^{a} (cospx + sinpx) dx$$

$$= 2\int_{0}^{a} cospx\, dx$$

$$= \frac{2sinap}{p} \quad p \neq 0$$

For $p = 0$ we find $f(p) = 2a$

Taking inverse fourier transform of $f(p)$ we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\,.\,e^{-ipx} dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2sinap}{p}\,.\,e^{-ipx} dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{sinap(cospx - sinpx)}{p} dp$$

$$= \begin{cases} 1 & for \quad |x| < a \\ 0 & for\ |x| > a \end{cases} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{sinapcospx}{p} dp$$

$$= \int_{-\infty}^{\infty} \frac{sinapcospx}{p} dp = \begin{cases} \pi & for \quad |x| < a \\ 0 & for\ |x| > a \end{cases} \text{--------(1)}$$

(ii) From (i)

$$\int_{-\infty}^{\infty} \frac{sinapcospx}{p} dp = \begin{cases} \pi & for \quad |x| < a \\ 0 & for\ |x| > a \end{cases}$$

$$\int_{0}^{\infty} \frac{sinapcospx}{p} dp = \begin{cases} \pi & for \quad |x| < a \\ 0 & for\ |x| > a \end{cases}$$

$$\int_0^\infty \frac{sinapcospx}{p}dp = \begin{cases} \pi/2 & for \ \ |x| < a \\ 0 & for\ |x| > a \end{cases}$$

Putting $x = 0\ and\ a = 1$ we get

$$\int_0^\infty \frac{sinapcos0}{p}dp = \frac{\pi}{2}$$

$$\int_0^\infty \frac{sinp}{p}dp = \frac{\pi}{2}$$

Example (4) Find the fourier cosine function of

$$f(x) = \begin{cases} x & for\ 0 < x < 1 \\ 2 - x & for\ 1 < x < 2 \\ 0 & for\ x > 2 \end{cases}$$

Solution : Fourier cosine transform of f(x)

$$F_c\{f(x)\} = \int_0^\infty f_c(x) cosxdx$$

$$\int_0^1 xcosx\ dx + \int_1^2 (2 - x)\ cosxdx + \int_2^\infty 0dx$$

$$\left[x.\frac{sinsx}{x} - \left(\frac{-cosx}{s^2}\right)\right]_0^1 + \left[(2 - x).\frac{sinsx}{x} - (-1)\left(\frac{-cosx}{s^2}\right)\right]_{10}^2$$

$$= \left(\frac{sins}{s} + \frac{coss}{s^2} - \frac{1}{s^2}\right) + \left(\frac{-cos2s}{s^2} - \frac{sins}{s} + \frac{coss}{s^2}\right)$$

$$= \frac{2coss}{s^2} - \frac{cos2s}{s^2} - \frac{1}{s^2}$$

Example(5) Find the fourier transform of

$$(x) = \begin{cases} 1 - x^2 & for \ \ |x| < a \\ 0 & for\ |x| > a \end{cases}$$

Hence evaluate $\int_0^\infty \frac{xcosx - sinx}{x^3} \cos\frac{x}{2}\ dx$

Solution: we know that fourier complex traosform

$$\boldsymbol{F}\{\boldsymbol{f}(\boldsymbol{x})\} = \int_{-\infty}^\infty f(x)e^{isx}dx = F(s)$$

$$= \int_{-\infty}^{-1} (\mathbf{0})\, e^{isx} dx + \int_{-1}^{1} (\mathbf{1} - \boldsymbol{x^2})\, e^{isx} dx + \int_{1}^{\infty} (\mathbf{0})\, e^{isx} dx$$

$$= \left[(\mathbf{1} - \boldsymbol{x^2}) \frac{e^{isx}}{is} - (\mathbf{2}\boldsymbol{x}) \frac{e^{isx}}{(\boldsymbol{is})^2} + (-\mathbf{2}) \frac{e^{isx}}{(is)^3} \right]_{-1}^{1} = \mathbf{2} \left(\frac{\boldsymbol{e^{is} + e^{-is}}}{\boldsymbol{-s^2}} \right) - \mathbf{2} \left(\frac{\boldsymbol{e^{is} - e^{-is}}}{\boldsymbol{-s^2}} \right)$$

Now by inversion formula we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx} ds$$

or $$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3} (scoss - sins)\, e^{-is/2} ds = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} \left(\frac{scoss - sins}{s^3} \right) \left(\cos\frac{s}{2} - isin\frac{s}{2} \right) ds = -\frac{3\pi}{8}$$

$$\int_{-\infty}^{\infty} \left(\frac{scoss - sins}{s^3} \right) \left(\cos\frac{s}{2} \right) ds = -\frac{3\pi}{8}$$

$$\int_{-\infty}^{\infty} \left(\frac{xcosx - sinx}{x^3} \right) \left(\cos\frac{x}{2} \right) dx = -\frac{3\pi}{16}$$

Example (6) Find fourier cosine transform of the following functions

(i) $$F(x) = \begin{cases} x & for\ 0 < x < 1/2 \\ 1 - x & for\ 1/2 < x < 1 \\ 0 & for\ x > 1 \end{cases}$$

(ii) $$F(x) = \begin{cases} cosx & for\ \ 0 < x < a \\ 0 & for\ x > a \end{cases}$$

(iii) $F(x) = e^{-2x} + 4e^{-3x}$

(iv) $F(x) = \frac{\cosh ax}{\cosh \pi x} = \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}}$

Solution: We have fourier cosine transform

(i) $f_c(p) = \int_0^\infty F(x) cospx dx$

$$= \int_0^{1/2} xcospx\, x + \int_{1/2}^{1} (1 - x) cospx dx$$

$$= \left(\frac{xsinpx}{p}\right)_0^{1/2} - \int_0^{\frac{1}{2}} \frac{sinpx}{p}\, dx + \left[(1-x)\frac{sinpx}{p}\right]_{\frac{1}{2}}^{1} - \int_{1/2}^{1} (-1)\frac{sinpx}{p} dx$$

$$= \frac{1}{2p}\sin\frac{p}{2} + \left(\frac{cospx}{p^2}\right)_0^{1/2} - \frac{1}{2p}\sin\frac{p}{2} - \left(\frac{cospx}{p^2}\right)_2^{1}$$

$$= \frac{1}{p^2}\left(\cos\frac{p}{2} - 1\right) - \frac{1}{p^2}(\cos p - \cos p/2)$$

$$= \frac{1}{p^2}\left(2\cos\frac{p}{2} - 1 - cosp\right)$$

(ii) We have fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) cospxdx$$

$$\int_0^a cosx.\, cospx\, dx = \frac{1}{2}\int_0^a 2cosx\, cospxdx$$

$$= \frac{1}{2}\int_0^a [\cos(1+p)\,x + \cos(1-p)\,x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(+-p)x}{(1+p)} - \frac{\sin(1-p)x}{(1-p)}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(1+p)a}{(1+p)} + \frac{\sin(1-p)a}{(1-p)}\right]$$

(iii) We have, fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) cospxdx$$

$$= \int_0^\infty (e^{-2x} + 4e^{-3x}) cospxdx$$

$$= \int_0^\infty e^{-2x} cospx\, dx + 4\int_0^\infty e^{-3x} cospx\, dx$$

$$= \frac{2}{p^2+4} + \frac{12}{p^2+9}$$

(iv) We have, fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) cospxdx$$

$$= \int_0^\infty \frac{\cosh ax}{\cosh \pi x}\ cospx\ dx$$

$$= \int_0^\infty \left(\frac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}\right) cospx\ dx$$

$$= \int_0^\infty \left(\frac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}\right)\left(\frac{e^{ipx}+e^{-ipx}}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_0^\infty \frac{e^{(a+ip)x}+e^{-(a+ip)x}}{e^{\pi x}+e^{-\pi x}}dx + \int_0^\infty \frac{e^{(a-ip)x}+e^{-(a-ip)x}}{e^{\pi x}+e^{-\pi x}}dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}sec\left(\frac{a+ip}{2}\right)+\frac{1}{2}sec\left(\frac{a-ip}{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{\cos(\frac{a+ip}{2})}+\frac{1}{\cos(\frac{a-ip}{2})}\right]$$

$$= \frac{1}{4}\frac{2\cos\frac{a}{2}\cos\frac{ip}{2}}{\cos(\frac{a+ip}{2})\cos(\frac{a-ip}{2})} = \frac{\cos\frac{a}{2}\cos\frac{ip}{2}}{cosa+coshp}$$

$$(cosp = coshp)$$

Example (6) Find the fourier cosine transform of

$f(x) = \frac{1}{1+x^2}$ hence derive fourier since transform of $\emptyset x = \frac{x}{x^2}$

Solution We have fourier cosine transform

$$f_c(p) = \int_0^\infty F(x)cospxdx$$

$= \int_0^\infty \frac{1}{1+x^2}cospxdx = I\ \ (let)$------------(1)

$\frac{dI}{ds} = \int_0^\infty \frac{-x\ sinx}{1+x^2}dx = -\int_0^\infty \frac{-x^2 sin\ sx}{x(1+x^2)}dx$------------(2)

$$= -\int_0^\infty \frac{[(1+x^2)-1]sin\ sx}{x(1+x^2)}dx$$

$$= -\int_0^\infty \frac{sin\ sx}{x}dx + \int_0^\infty \frac{sin\ sx}{x(1+x^2)}dx$$

$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{sin\ sx}{x(1+x^2)}dx$-------------(3)

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)} ds = I$$

$$\frac{d^2I}{ds^2} - I = 0$$

$(D^2 - 1)I = 0$ where $D = \frac{dI}{ds}$

It's solution (c.f) $I = c_1e^s + c_2e^{-s}$-----------(4)

$\frac{dI}{ds} = c_1e^s - c_2e^{-s}$-----------(5)

Where $s = 0$ (i) snd (iv) given

$c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$---------(6)

Also where $s = 0$ (iii) and (v) given

$c_1 - c_2 = -\frac{\pi}{2}$--------------(7)

Solving (6)and(7) we get

$$c_1 = 0, \;\; c_2 = \frac{\pi}{2}$$

From (4) $I = c_1e^s + c_2e^{-s}$

$$= 0 + \pi/2e^{-s}$$

$$I = \pi/2e^{-s}$$

From (1) $I = \int_0^\infty \frac{\cos px}{(1+x^2)} dx = -\pi/2e^{-s}$

Diff write to we get

$$\int_0^\infty \frac{-x\sin sx}{(1+x^2)} dx = -\pi/2e^{-s}$$

$$\int_0^\infty \frac{x\sin sx}{(1+x^2)} dx = \pi/2e^{-s}$$

Example: (7) Solve the integral equation

$$\int_0^\infty F(x)cospxdx = \begin{cases} 1-p & for \quad 0 \le p \le 1 \\ 0 & for\, p \ge 1 \end{cases}$$

Hence proved that $\int_0^\infty \frac{sin^2 t}{t^2}\, dt = \pi/2$

Solution: We have fourier cosine transform

$$f_c(p) = \int_0^\infty F(x)cospxdx$$

Then $f_c(p) = \begin{cases} 1-p & for \quad 0 \le p \le 1 \\ 0 & for\, p \ge 1 \end{cases}$

By inversion formula for fourier cosine transform, we have

$$F(x) = \frac{2}{\pi}\int_0^\infty f_c(p)\cos px\, dp$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-p)cospxdp + \int_1^\infty 0.\, cospx\, dp\right]$$

$$= \frac{2}{\pi}\left[(1-p)\frac{sinpx}{x} - (-1)\frac{-cospx}{x^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-cosx}{x^2} + \frac{1}{x^2}\right] = \frac{2}{\pi x^2}\ (1-cosx)$$

To proof

Since $\int_0^\infty F(x)cospxdx = \begin{cases} 1-p & for \quad 0 \le p \le 1 \\ p & for\, p \ge 1 \end{cases}$

Where $F(x) = \frac{2(1-cosx)}{\pi x^2}$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{1-cosx}{x^2}\right) cospx\, dx = \begin{cases} 1-p & for \quad 0 \le p \le 1 \\ p & for\, p \ge 1 \end{cases}$$

When $p = 0$ we have $\frac{2}{\pi}\int_0^\infty \frac{1-cosx}{x^2} dx = 1$

or $\int_0^\infty \frac{2sin^2\frac{x}{2}}{x^2}\, dx = \pi/2$

(Put $x = 2t, dx = 2dt$) $\int_0^\infty \frac{sin^2 t}{t^2} dt = \frac{\pi}{2}$

Example Using parseval's identities proved that

(i) $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$

(ii) $\int_0^\infty \frac{t^2}{(t^2+1)^2} dt = \frac{\pi}{4}$

(iii) $\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$

(iv) $\int_0^\infty \frac{sinat}{t(a^2+t^2)} dt = \frac{\pi}{2} \frac{1-e^{-a^2}}{a^2}$

Solution:

(i) Let$F(x) = e^{-x}$ so that $F_c(s) = \frac{1}{1+s^2}$

By parseval's identity for cosine transform

$$\frac{2}{\pi}\int_0^\infty [F_c(s)]^2 ds = \int_0^\infty |f(x)|^2 dx$$

$$\frac{2}{\pi}\int_0^\infty \left|\frac{1}{(1+s^2)^2}\right|^2 ds = \int_0^\infty |e^{-x}|^2 dx$$

$$= \int_0^\infty e^{-2x} dx = \left[\frac{e^{-2x}}{e^2}\right]_0^\infty$$

Hence $\int_0^\infty \left|\frac{1}{(1+s^2)}\right|^2 ds = \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}$

(ii) Let $F(x) = \frac{x}{x^2+1}$ so that $F_c[f(x)] = \frac{\pi}{2}e^{-s}$

Now using parseval's identity for transform i.e

$$\frac{2}{\pi}\int_0^\infty [F_s f(x)]^2 dx = \int_0^\infty [f(x)]^2 dx$$

$$\int_0^\infty \left(\frac{x}{x^2+1}\right)^2 dx = \frac{2}{\pi}\int_0^\infty \left(\frac{\pi}{2}e^{-s}\right)^2 ds$$

$$= \frac{\pi}{2}\left[\frac{e^{-2s}}{-2}\right]_0^\infty = \frac{\pi}{4}(0-1) = \frac{\pi}{4}$$

Hence

$$\int_0^\infty \left(\frac{x}{x^2+1}\right)^2 dx = \frac{\pi}{4}$$

Replace x by t

$$\int_0^\infty \left(\frac{t}{t^2+1}\right)^2 dt = \frac{\pi}{4}$$

(iii) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

Then $F_c(s) = \frac{a}{a^2+s^2}$ and $G_c(s) = \frac{b}{b^2+s^2}$

Now using parseval's identity for forier cosine transform i.e

$\frac{2}{\pi}\int_0^\infty F_c(s)\, G_c(s) ds = \int_0^\infty f(x) g(x) dx$----------(1)

We have

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)} = \int_0^\infty e^{-(a+b)x} dx$$

or

$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{1}{a+b}$\

Then

$$\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab(a+b)}$$

Replace s by t

$$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(iv) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1 & for \quad 0 < x < a \\ 0 & for\ x > 1 \end{cases}$

Then $F_c(s) = \frac{a}{a^2+s^2}$ and $G_c(s) = \frac{b}{b^2+s^2}$

Now using parsval's identity for fourier cosine transform i.e

$$\frac{2}{\pi}\int_0^\infty F_c(s)\, G_c(s) ds = \int_0^\infty f(x) g(x) dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a\ sinas}{s(a^2+s^2)} ds = \int_0^\infty e^{-ax} dx$$

$$= \frac{1 - e^{-ax}}{a}$$

$t$hen $\int_0^{\infty} \frac{sinat}{t(a^2+t^2)} dt = \frac{\pi}{2a^2}(1 - e^{-a^2})$

Example find fourier sine and cosine transform

(i) $x^{m-1}\backslash$

(ii) x^{-m}

Solution We $F(x) = x^n e^{-ax}$

$$f_s(p) = \int_0^{\infty} x^n e^{-ax} sinpx\, dx$$

$$= \int_0^{\infty} x^n e^{-ax} \left(\frac{e^{ipx} - e^{-ipx}}{2i}\right) dx$$

$$= \frac{1}{2i}\left[\int_0^{\infty} x^n e^{-(a-ip)x} dx - \int_0^{\infty} x^n e^{-(a+ip)x} dx\right]$$

$$f_s(p) = \frac{1}{2i}\left[\frac{\sqrt{n+1}}{(a-ip)^{n+1}} - \frac{\sqrt{n+1}}{(a+ip)^{n+1}}\right] \text{-----------(1)}$$

Using Gamma function $\int_0^{\infty} e^{-zx} x^{n-1}\, dx = \frac{\sqrt{n}}{z^n}$

Put $a = rcos\theta,\ p = r\ sin\theta$

So that $(a + ip)^{n+1} = r^{n+1}\{\cos(n+1)\,\theta + isin(n+1)\theta\}$

and $a - ip = r(cos\theta - isin\theta)$

$$(a - ip)^{n+1} = r^{n\mp 1}\{\cos(n+1)\theta - isin(n+1)\theta\}$$

Also $r^2 = a^2 + p^2$ and $tan\theta = b/a$

Using (1) Now $F_c(p) = \frac{\sqrt{n+1}}{2i}\left[\frac{2ir^{n+1}\sin(n+1)\theta}{r^{2(n+1)}}\right]$

$$= \frac{\sqrt{n+1}\sin(n+1)\theta}{r^{(n+1)}} = \frac{\sqrt{n+1}\sin(n+1)\theta}{(a^2+p^2)^{n+1/2}}$$

Simply $f_s(p) = \frac{\sqrt{n+1}\cos(n+1)\theta}{(a^2+p^2)^{n+1/2}}$

We have the relations

$$\int_0^\infty e^{-ax}.x^n sinpx\,dx = \frac{\sqrt{n+1}\sin(n+1)\theta}{(a^2+p^2)^{n+1/2}}$$-----------(2)

And

$$\int_0^\infty e^{-ax}.x^n cospx\,dx = \frac{\sqrt{n+1}\cos(n+1)\theta}{(a^2+p^2)^{n+1/2}}$$-----------(3)

Where$tan\theta = b/a$

Put $a = 0$ and replace n by $(m-1)$ in (2) and (3)

We get $\int_0^\infty x^{m-1} sinpx\,dx = \frac{\sqrt{m}sin\frac{m\pi}{2}}{p^m}$-----------(4)

and $\int_0^\infty x^{m-1} cospx\,dx = \frac{\sqrt{m}cos\frac{m\pi}{2}}{p^m}$-----------(5)

(iii) Replace m by (m+1)in (4) and (5) we get

$\int_0^\infty x^m sinpx\,dx = \frac{\sqrt{m+1}sin\frac{(m+1)\pi}{2}}{p^{m+1}} = \frac{-\sqrt{m+1}cos\frac{m\pi}{2}}{p^{m+1}}$---------(6)

And $\int_0^\infty x^m cospx\,dx = \frac{-\sqrt{m+1}sin\frac{m\pi}{2}}{p^{m+1}}$---------(7)

Replacing m by (-m) in (6) $ (7) we get

$$\int_0^\infty x^{-m} sinpx\,dx = \frac{\sqrt{1-m}sin\frac{m\pi}{2}}{p^{1-m}}$$

$$\frac{\pi}{sinm\pi}.\frac{1}{\sqrt{m}}\frac{sin\frac{m\pi}{2}}{p^{1-m}}$$

$$= \frac{p^{1-m}}{\sqrt{m}}\frac{\pi sin\frac{m\pi}{2}}{2sin\frac{m\pi}{2}cos\frac{m\pi}{2}} = \frac{\pi}{2}\frac{p^{m-1}}{\sqrt{m}}sec\frac{m\pi}{2}$$

$$\int_0^\infty x^{-m} cospx\,dx = \frac{\pi}{2}\frac{p^{m-1}}{\sqrt{m}}cosec\frac{m\pi}{2}$$

Exercise ( )

(1) Find fourier transform of

$$f(x) = \begin{cases} 0 & for \quad x < a \\ 1 & for \ a < x < b \\ 0 & for \ x > b \end{cases}$$

(2) Find fourier transform of

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

hence evaluate $\int_0^\infty \frac{sinx}{x}\, dx$

(3) Find the function if its sin transform in $\frac{e^{-s}}{s}$ Ans $\sqrt{\frac{\pi}{2}}$

(4) Using parseval's identity evaluate

$\int_0^\infty \left(\frac{1-cosx}{x}\right)^2 dx$ Ans $\frac{\pi}{2}$

(5) Using parseval's identity prove

$$\int_0^\infty \left(\frac{sint}{t}\right)^2 dt = \frac{\pi}{2}$$

(6) Find the fourier's cosine integral of the function e^{-ax}. Hence Show that

$\int_0^\infty \frac{cos2x}{\lambda^2+1^2} d\lambda = \frac{\pi}{2} e^{-x}$, $x \geq 0$

(7) Using fourier integral's show that

$$\int_0^\infty \frac{\lambda sin\lambda x}{\lambda^2 + k^\lambda} d\lambda = \frac{\pi}{2} e^{-kx}$$

(8) Find fourier cosine transform of $e^{-a^2x^2}$ and hence evaluate sine transform $xe^{-a^2x^2}$

(9) Find the fourier cosine transformed

$$f(x) = e^{-2x} + 4e^{-3x}$$

(10) Find fourier sine transform of $\frac{e^{-ax}}{x}$. Hence find sine transform of $\frac{1}{x}$

FOURIER TRANSFORMS OF DERIVATIVES OF A FUNCTION

We have

$$F\{f^n(x)\} = (-is)^n F(s)$$

The fourier transform of the function let (x,t) is given by

$F[\mu(x,t)] = \int_{-\infty}^{\infty} \mu(x,t) e^{ipx} dx0$

Suppose μ and $\frac{\partial \mu}{\partial x}$ both varish as $x \to \pm\infty$ then the fouriertransform of $\frac{\partial^2 y}{\partial x^2}$-

$$F\left[\frac{\partial^2 y}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2} e^{ipx} dx$$

$$= \left[\left\{e^{ipx}\frac{\partial \mu}{\partial x} - ipe^{ipx}.u\right\}_{-\infty}^{\infty} + \int_{-\infty}^{\infty} ip^2 e^{ipx}.\mu dx\right]$$

$$= -p^2 \int_{-\infty}^{\infty} \mu e^{ipx} dx = -p^2 \bar{\mu}$$

Hence $$F\left[\frac{\partial^2 y}{\partial x^2}\right] = -p^2\bar{\mu} = -p^2 F(\mu)$$

Similarly in case of fourier sine and cosine transform we get

$$F_s\left[\frac{\partial^2 y}{\partial x^2}\right] = p(\mu) - p^2\bar{\mu}_s$$
$$= p(\mu) - p^2 F_s(\mu)$$

And

$$F_c\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{\partial y}{\partial x} = -p^2 F_c(\mu) = -\frac{\partial y}{\partial x} - p^2\bar{\mu}_s$$

In genral, the fourier transform of n^{th} of $f(x) is\ given\ by$

$$F\frac{d^n F}{dx^n} = (-ip)^n F[f(x)]$$

CHOICE OF INDFINITE FOURIER SINE OR COSINE TRANSFORM

For exclusion of $\frac{\partial^2 y}{\partial x^2}$ from a differential equation we have

(i) $(\mu)_{x=0}$ in sine transform

(ii) $(\frac{\partial \mu}{\partial x})_{x=0}$in cosine transform

Relation between fourier laplace transform

Let $f(x) = \begin{cases} e^{-st} g(t) & for \quad t > 0 \\ 0 & for \quad t < 0 \end{cases}$

Then the fourier transform of $f(t)$ in given by

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ist} f(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(is-x)t} g(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-pt} g(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} L\,\{g(t)\}$$

$Fourier\ transform = \frac{1}{\sqrt{2\pi}}$ ×laplace tyransform $g(t)$

Application of fourier transform

In one dimensional boundary value problems, the portial differential equation can earily transform ed into an ordinary differential equation by applying is than obtained by solving this equation and inviting by means of the complex inversion fourier or by any other method.

In two dimasional problem, it is som time required to apply two transform twice and the desired solution obtained by double inversion.

Step(1) In a problem $\mu(x,t)x = 0$ is given then we use infinite sine transform to remove $\frac{\partial^2 y}{\partial x^2}$ from the differential equation.

Step(2) In case $\left[\frac{\partial u(x,t)}{\partial x}\right] x = 0$ is given then we employ infinite cosine transform to remove $\frac{\partial^2 y}{\partial x^2}$

Step(3) If in a problem $\mu(0,t)$ and $\mu(l,t)$ are given then we use finite sine transform to remove $\frac{\partial^2 y}{\partial x^2}$from the deferential eqution.

Step(4) In case $(\frac{\partial u}{\partial x})_{x=0}$ and $(\frac{\partial u}{\partial x})_{x=l}$ are given then we empolyfinite cosen transform to remove $\frac{\partial^2 y}{\partial x^2}$

One dimensional Heat transform Equation

Example ( ) Solve $\frac{\partial u}{\partial x} = k\frac{\partial^2 y}{\partial x^2}$ for $0 \leq x < \infty,\ t > 0$ given the condition

(i) $\mu_{(x,0)}$ For$x \geq 0$ (ii) $(\frac{\partial u}{\partial x})_{(0,t)} = -a$(constant)

(iii) $\mu_{(x,t)}$ is bounded.

Solution In this problem $\frac{\partial u}{\partial x}$ at $x = 0$given hence take fourier cosine transform or both sides of the given equation

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(k\frac{\partial^2 y}{\partial x^2}\right)$$

$$\frac{\partial \bar{u}}{\partial t} = k\left[(-s^2\bar{\mu}) - \sqrt{\frac{2}{\pi}}\frac{\partial u}{\partial x}(0,t)\right]$$

$$= -ks^2\bar{\mu} + \sqrt{\frac{2}{\pi}}\,\text{ka}$$

this is linear in $\bar{\mu}$ therefore solving $\bar{\mu}e^{ks^2t} = \int \sqrt{\frac{2}{\pi}}\ kae^{ks^2t}\ dt$

$$\bar{\mu}e^{ks^2t} = \sqrt{\frac{2}{\pi}}ka.\frac{e^{ks^2t}}{ks^2} + c$$

$$\bar{\mu}(s,t) = \sqrt{\frac{2}{\pi}}\frac{a}{s^2} + ce^{-ks^2t}\text{----------(1)}$$

Since $\mu(x,0) = 0$ (given) for $x \geq 0$

$$\bar{\mu}(s,0) = 0$$

Using this eq (1)

$$\bar{\mu}(s,0) = c + \sqrt{\frac{2}{\pi}}\frac{a}{s^2} = 0$$

$$\boldsymbol{c} = -\sqrt{\frac{2}{\pi}}\frac{a}{s^2}$$

substating ths in (1)

$$\bar{\mu}(s,t) = \sqrt{\frac{2}{\pi}}\frac{a}{s^2}(1 - e^{-ks^2t})$$

By inversion thermo

$$\mu(x,t) = \frac{2}{\pi}.a\int_0^\infty \frac{1 - e^{-ks^2t}}{s^2}\cos sx\, ds$$

Example ( ) Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 y}{\partial x^2}(x > 0, t > 0)$ subject to the conditions.

(i) $\mu = 0$ when $x = 0, t > 0$

(ii) $\mu = \begin{cases} 1 & 0 < x > t \\ 0 & x \geq 1 \quad when\ t = 0 \end{cases}$

(iii) $\mu(x,t)$ in banded

Solution Since $\mu(0,t)0$, we take fourier sine transform of both sides of the equation we get

$$\int_0^\infty \frac{\partial u}{\partial t}\ sinsx\ dx = \int_0^\infty \frac{\partial^2 y}{\partial x^2}\sin sx\ dx$$

$$\frac{\partial}{\partial t}\int_0^\infty \mu\sin sx\ dx = -s^2\bar{\mu} + s\mu(0)$$

$$\frac{\partial \bar{\mu}}{\partial t} = -s^2\bar{\mu}$$

$$\frac{\partial \bar{\mu}}{\partial t} + s^2\bar{\mu} = 0$$

$(D^2 + s^2)\bar{\mu} = 0$ i.e $d = \pm is$

Its solution is $\bar{\mu}(s,t) = e^{-s^2t}$---------------------(1)

Since $\bar{\mu}(s,t) = \int_0^\infty \mu(x,t)sinsx\ dx$

$$= \frac{1-soc\ s}{s} \text{-------------------------}(2)$$

From (1) and (2) we get

$$c = \bar{\mu}(s,0) = \frac{1-soc\ s}{s}$$

Thus (1) given $\bar{\mu}(s,t) = \frac{1-soc\ s}{s} e^{-s^2 t}$

Now taking fourier sine transform we get

$$\mu(x,t) = \int_0^\infty \frac{1-soc\ s}{s} e^{-s^2 t}\, ds$$

which is deride solution

Example ( ) Use fourier sine transform to Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 y}{\partial x^2}$

Under the conditions.

(i) $\mu(0,t) = 0$
(ii) $\mu(x,0) = e^{-x}$
(iii) $\mu(x,t)$ in banded

Solution The given equation is

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 y}{\partial x^2} \text{----------}(1)$$

Taking fourier sine transform on both of equation we get

$$\int_0^\infty \frac{\partial u}{\partial t}\ sin\ px\ dx = 2\int_0^\infty \frac{\partial^2 y}{\partial x^2} \sin px\ dx$$

$$\frac{\partial \bar{\mu}}{\partial t} = 2[p(\mu) - p^2\bar{\mu}]$$

$$\frac{\partial \bar{\mu}}{\partial t} + 2p^2\bar{\mu} = 0 \text{ ----------(where} \bar{\mu} \int_0^\infty \mu sin\ px\ dx\ )$$

Its solution is

$$\mu = c_1 e^{-2p^2 t} \text{------------}(2)$$

where c_1 is constant

At t=0

$$(\bar{\mu}) = \int_0^{\infty} \mu_{t=0} sin\ px\ dx = \int_0^{\infty} e^{-x} sinpx\ dx$$

$$= \frac{p}{1+p^2} \text{--------------(3)}$$

$$\frac{\partial \bar{\mu}}{\partial t} + c^2 p^2 \bar{\mu} = 0 \text{-------------(4)}$$

Solution of (4) is $(D^2 + c^2 p^2)\bar{\mu} = 0$

$$\bar{\mu} = c_1 e^{-c^2 p^2 t} \text{-------------(5)}$$

Taking sine transform of (2)we get

$$(\bar{\mu}) = \int_0^{\infty} F(x) sin\ px\ dx$$

$$= f_s(p)$$

From(5)

$$(\bar{\mu}) = c_1 = f_s(p)$$

From

$$(\bar{\mu}) = f_s(p) e^{-c^2 p^2 t}$$

Now taking into inverse fourier sine transform we get

$$\mu(x,t) = \frac{2}{\pi} \int_0^{\infty} f_s(p) e^{-c^2 p^2 t} \sin px\ dp$$

One dimensional Wave Equation

Eaample (   ) An infinity long string having one at $x = 0$ in infitaly at rest along $x -$ axis. The end $x = 0$ is given a transverse displacement . When $t > 0.$ Find the displacement of any point of the string.

Solution Let $y(x,t)$ be the displacement. Then we have

Equation is

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2} \text{----------(1)}$$

Subject to the conditions-

$y(x,0) = 0$-------------(2) $\quad \frac{\partial y}{\partial t}_{(x,0)} = 0$ -----------(3)

$y(0,t) = f(t)$ ----------(4) $\quad y(x,t)$ is banded --------(5)

On taking Laplace transform (1) we get

$$L\left(\frac{\partial^2 y}{\partial x^2}\right) = c^2 L \frac{\partial^2 y}{\partial t^2}$$

$$s^2 \bar{y} = -sy(x,0) - \frac{\partial y}{\partial t}_{(x,0)} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \text{-----------(7)}$$

On putting $y(x,0) = 0$, $\frac{\partial y}{\partial t}_{(x,0)} = 0$ in (6) we get

$$s^2 \bar{y} = c^2 \frac{\partial^2 \bar{y}}{\partial t^2} \rightarrow \frac{\partial^2 \bar{y}}{\partial x^2} = \left(\frac{s}{c}\right)^2 \bar{y} \text{ ----------(7)}$$

Laplace transform of (4)

$$\bar{y}(0,s) = \bar{f}(s) \;\; at\; x = 0 \text{ ---------(8)}$$

On solving (7) we get

$$\bar{y} = Ae^{\frac{sx}{c}} + Be^{-\frac{sx}{c}} \text{ --------------------(9)}$$

According to condition (5) y is finited at $x \rightarrow \infty$ this given $A = 0$

$$\bar{y} = Be^{-\frac{sx}{c}} \text{ ----------------------------(10)}$$

Putting the valu of $\bar{y}(0,s) = \bar{f}(s)$ at $x = 0$ in (10) we get

$\bar{f}(s) = B$

then (10) becom $y = \bar{f}(s)e^{-\frac{sx}{c}}$

To get y from $\bar{y}$, we use complex inversion formula

$$y = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\left(t-\frac{x}{c}\right)s} - f(s)ds$$

$$y = f(t - x/c)$$

Eaample (   ) An infinite string is initially at rest and that the intial displacement is $f(x)$ $(-\infty < x < \infty)$. Determine the displacement $y(x,t)$ of the string.

Solution the eqution for vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\text{----------(i)}$$

and the intial condition are

$$\frac{\partial y}{\partial t}_{(t=0)} = 0 \qquad y(x,0) = f(x)\text{---------(ii)}$$

Multiplying (i) by e^{isx} and integral w.r to x from $-\infty$ to ∞ we get

$$\frac{\partial^2 y}{\partial t^2} = c^2(-s^2 y)$$

Asolution of

$$\frac{d^2 y}{dt^2} + c^2 s^2 y = 0$$

Y=$A_1 \cos cst + A_2 \sin cst$-----------(3)

Also forier transform of (ii) are

$\frac{\partial y}{\partial t}_{(t=0)} = 0$ and $y = F_s$ when t=0

Appliying these to (3) we get

$A_2 = 0$ and $A_1 = F(s)$

Thus

$y = F(s)\cos cst$

Now taking inverse fourier transform we get

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cos cst\,.\,e^{-isx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s).\left(\frac{e^{ict}+e^{-cst}}{2}\right)e^{-isx}\,dx$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}[F(c)e^{-is(x-ct)} + F(s)e^{-is(x+ct)}]ds$$

$$= \frac{1}{2}[f(x-ct) + f(x+ct)]$$

Transmissio Lines

Example ( ) A semi- infinite transmission line of negligible inductance and leaknce per unit length has it voltage and current equal to zero. A constant voltage v_0 in applied at the sending end $(x = 0)$ at $t = 0$ find the voltage and current at point $(x > 0)$ and at any instant.

Solution: Let $v(x,t)$ and $i(x,t)$ be the voltage and cuuent at any point x and at any time t if $L = 0$ and $G = 0$ then the trans mission line equation becomes

$$\frac{\partial \gamma}{\partial x} = -Ri, \qquad \frac{\partial i}{\partial x} = -c\frac{\partial \gamma}{\partial t}$$

i.e $\frac{\partial^2 v}{\partial x^2} = Rc\frac{\partial \gamma}{\partial t}$ ---------(i)

The bandry conditions are $\gamma(0,t) = \gamma_0$ and $i(x,t)$ in finite for all x and t

The intial condition one $\gamma(x,0) = 0$, i($x,0$)$= 0$-----------(ii)

Laplace transform of (i) are-

$$\frac{\partial^2 \gamma}{\partial x^2} = Rc(s\bar{v} - 0)$$

Or $$\frac{\partial^2 \gamma}{\partial x^2} = Rcs\bar{v} = 0$$ ----------(iii)

Laplace transform of the condition (ii) are

$$\bar{\gamma}(0,s) = \frac{\gamma_0}{s} \text{ at } x = 0$$ -------(iv)

And $\bar{\gamma}(x,s)$ remain finite as $x \to 0$ ------(v)

The solution of (iii) in

$\bar{\gamma}(x,s) = c_1 e^{\sqrt{Rcs}x} + c_2 e^{-\sqrt{Rcs}x}$

To satisfy condition (v) we mul have $c_1 = 0$

Using the condition (iv) we get $c_2 = \frac{v_0}{s}$

Then $\bar{\gamma}(x,s) = \frac{v_0}{s}\, e^{-\sqrt{Rcs}x}$

Using the inversion formula we obtain

$$v(x,t) = v_0 L^{-1}\left\{\frac{e^{-\sqrt{Rc}x\sqrt{s}}}{s}\right\}$$

$$v_0 \frac{x\sqrt{Rc}}{2\sqrt{\pi}} \int_0^t \mu^{-3/2} \quad e^{-\left(\frac{Rcx^2}{4u}\right)} du$$

Since $i = -\frac{1}{R}\frac{\partial v}{\partial x}$ we obtain by differentiation

$$i(x,t) = \frac{v_0 x}{2\sqrt{x}} \sqrt{\frac{c}{R} t^{-3/2} e^{-\left(\frac{Rcx^2}{4u}\right)}}$$

Exercise

(1) The temperature μ in the semi- infinite rod $0 \le x < \infty$ is determined by the deferential equation

$$\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{dt^2} \text{ subject to condition}$$

(i) $\mu = 0$ when $t = 0, x \ge 0$

(ii) $\frac{\partial u}{\partial x} = -\mu$ When $x = 0$ and t >0

Making use of cosine transform , show that

$$\mu(x,t) = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos px}{p^2}(1 - e^{-kr^2t})dp$$

(2) Apply forier transform to solve the PDE

$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 v}{dt^2}\; x > 0, t > 0$$

subject to condition

(i) $V_x(0,t) = 0$

(ii) $V(x,0) = \begin{cases} x, & 0 \le x < 1 \\ 0 & x > 1 \end{cases}$

(iii) $V(x,t)$is bounded

(3) If the initial temperature of infinite bar is given by

$$\mu(x,0) = \begin{cases} 1 & for\ -c < x < c \\ 0 & for\ otherwise \end{cases}$$

Determine the temperature of an infinite bar at any point x and at any time $t > 0$

(4) A tighty stretched string with fixed end point $x = b$ and $x = c$ is inifially in a pointion given by $y = bsin\left(\frac{\pi x}{c}\right)$. It is released from rest in this position. Show by the method of Laplace transform that the displacement y at any distance x from one end and at any time t is given by

$y = bsin\frac{\pi x}{c}cos\frac{\pi q}{c}t$ and y satisfies the equation $\frac{\partial^2 y}{dt^2} = a^2\frac{\partial^2 y}{dt^2}$

(5) Solve the equation for high voltage semi infinite line with the following intial and boundary conditions-

$\gamma(x,t) = 0$ and $i(x,0) = 0$

$\gamma(0,t) = V_0\mu(t)$

$V(x,t)$ is finite as $x \to \infty$

FINITE FOURIER TRANSFORMS

Let $f(x)$ denote a function which is section ally continuous over the range $(0,l)$

Fourier line transform of $f(x)$ on this inteveal is defined as

$$F_s(p) = \overline{f_s}(p) = \int_0^l f(x).sin\frac{P\pi x}{l}dx$$

When p is an integer (instead of s, we take$p$ as a parameter)

Inversion formula for sine transform

If $f_s(p) = F_s(p)$ is the finite fourier sine transform of $f(x)$ in $(0, l)$ then inversion formula for sine transform is $f(x) = \frac{2}{l}\sum_{p=l}^{\infty}\overline{f_s}(p)sin\frac{P\pi x}{l}$

Finite Fourier cosine transform

Let $f(x)$ denote a section ally continuous function in $(0, l)$

Then the finite fourier cosine treansform of $f(x)$ over $(0, e)$ is depined as

$F_c(p) = \overline{f_c}(p) = \int_0^l f(x)cos\frac{p\pi x}{l}dx$ where p is an integer

Inversion formula for cosine transform

If $\overline{f_c}(p)$is the finite fourier cosine transform of F_x in $(0, l)$ then the inversion formula for cosine transform is $f(x) = \frac{1}{l}\overline{f_c}(0) + \frac{2}{l}\sum_{p=l}^{\infty}\overline{f_c}(p)cos\frac{p\pi x}{l}$

Where $\overline{f_c}(0) = \int_0^l f(x)dx$

Example find the five fourier sine and cosine transform

(i) $f(x) = 1$ in $(0, \pi)$

(ii) $f(x) = x^3$ $in\ (0, e)$

Solution

(i) $\overline{f_c}(p) = \overline{F_c}(1) = \int_0^\pi 1.sin\frac{p\pi x}{\pi}dx$

$$= \left(-\frac{cospx}{p}\right)_0^\pi = \frac{1 - cosp\pi}{p}dx$$

(ii) $\overline{F_c}(x^3) = \int_0^l x^3\ sin\frac{p\pi x}{\pi}dx$

$$= \left[x^3\left\{-\frac{cosp\pi x/l}{p\pi/l}\right\} - (-3x^2)\left\{\frac{sinp\pi x/l}{p^2\pi^2/l^2}\right\}\right]$$

$$= \frac{-l^p}{p\pi}(-1)^p + \frac{6l^4}{p^3\pi^3}(-1)^p$$

$F_c(x^3) = \int_0^1 x^3 cos\,\pi x/l$ dx

$$\left[\left(x^3\left\{\frac{\frac{sinp\pi x}{l}}{\frac{p\pi x}{l}}\right\}\right) - (-3x^2)\left\{\frac{\frac{cosp\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right\} + 6x\left(\frac{\frac{sinp\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right) - 6\left(\frac{\frac{cosp\pi x}{l}}{\frac{p^4\pi^4}{l^4}}\right)\right]$$

$$= \frac{3l^4}{p^2\pi^2}(-1)^p - \frac{6l^4}{p^4\pi^4}[-1^p - 1]$$

Example ( ) Using finite fourier transform, solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{dx^2} \qquad \text{give } \mu(0,t) = 0 \text{ and } \mu(4,t) = 0 \text{ and } \mu(x,t) = 2x$$

Solution Since $\mu(0,t) = 0$ given take finite fourier cosine transform-

$$\int_0^4 \frac{\partial u}{\partial t} sin\frac{p\pi x}{4} dx = \int_0^4 \frac{\partial^2 u}{dx^2} sin\frac{p\pi x}{4} dx$$

$$\frac{d}{dt}\overline{U_s} = \bar{f_s}\left(\frac{\partial^2 u}{dx^2}\right)$$

$$= -\frac{p^2\pi^2}{16}\bar{\mu_s} + \frac{p\pi}{4}[u(0,t) - (-1)\mu(4,t)]$$

$$\frac{p^2\pi^2}{16}\bar{\mu_s}$$

$$\frac{\partial\bar{\mu_s}}{\mu_s} = \frac{p^2\pi^2}{16}dt$$

Integral $log\bar{\mu_s} = -\frac{p^2\pi^2}{16} + c$

$$\bar{\mu_s} = Ae^{-\frac{p^2\pi^2}{16}t}$$

Since $\mu(x, 0 = 2x)$

$$\bar{\mu_s}(p,0) = \int_0^4 (2x)sin\left(\frac{p\pi x}{4}\right)dx$$

$$= \frac{32}{p\pi} cosp\pi \textbf{-----------(2)}$$

Substiting in (i)

$$\bar{\mu_s} = \frac{32}{p\pi}(-1)e^{-\frac{p^2\pi^2}{16}t}$$

By inversion theorem

$$\mu(x,t) = \frac{4}{4}\sum_{p=1}^{\infty}\frac{32}{p\pi}(-1)e^{-\frac{p^2\pi^2}{16}t}$$

Example Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{dx^2}$,$0 < x < 4$,$t > 0$ given

$\mu(0,t) = 0$ and $\mu(4,t) = 0$

$\mu(x,0) = 2sin\pi x - 2sins\pi x$

Solution Since $\mu(0,t)$is given take finite fourier sine transform. The equations

$$\frac{d\bar{\mu_s}}{dt} = 2\left[-\frac{p^2\pi^2}{16}\bar{\mu_s} + \frac{p\pi}{4}\{\mu(0,t) - (-1)^p\mu(4,t)\}\right]$$

$$= \frac{-p^2\pi^2}{8}\bar{\mu_s}$$

Solving we get

$$\bar{\mu_s} = Ae^{-\frac{p^2\pi^2}{8}t}\text{---------(i)}$$

Given $\mu(x,0) = 3sin\pi x - 2sin5\pi x$

Taking sine transform

$$\bar{\mu_s}(p,0) = \int_0^4 (3sin\pi x - 2sin5\pi x)\, sin\frac{p\pi x}{4}\, dx$$

$= 0$ if $p \neq 4, p \neq 0$

If $p = 4$ $\bar{\mu_s}(4,0) = 6$

If $p = 0$ $\bar{\mu_s}(20,0) - 46$

$$\mu(x,t) = \frac{2}{4}\sum_{p=1}^{\infty}\bar{\mu_s}(p,t)sin\left(\frac{p\pi x}{4}\right)$$

$$= \frac{1}{2}\left[6e^{-\frac{p^2\pi^2}{8}t}sin\pi x - 4e^{-\frac{p^2\pi^2}{8}t}sin5\pi x\right]$$

Where p in the first term is 4 and p is the second term is 20

$$= 3e^{-2\pi^2 t}sin\pi x - 2e^{-50\pi^2 t}sin5\pi x$$

Exercise

(1) Find the finite fourier sine and cosine transform of $f(x) = 2x$ in $(0,4)$

(2) Find the finite fourier sine and cosine transform of $f(x) = cosax$ in $(0,\pi)$

(3) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{dx^2}$

Given $\frac{\partial u}{\partial t}(0,t) = 0$

$\frac{\partial u}{\partial x}(6,t) = 0$ and $u(x,0) = 2x$

(4) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{dx^2}$

$v(0,t) = 1$

$v(\pi,t) = 3$

$v(x,0) = 1$ for $0 < x < \pi \;\; t > 0$

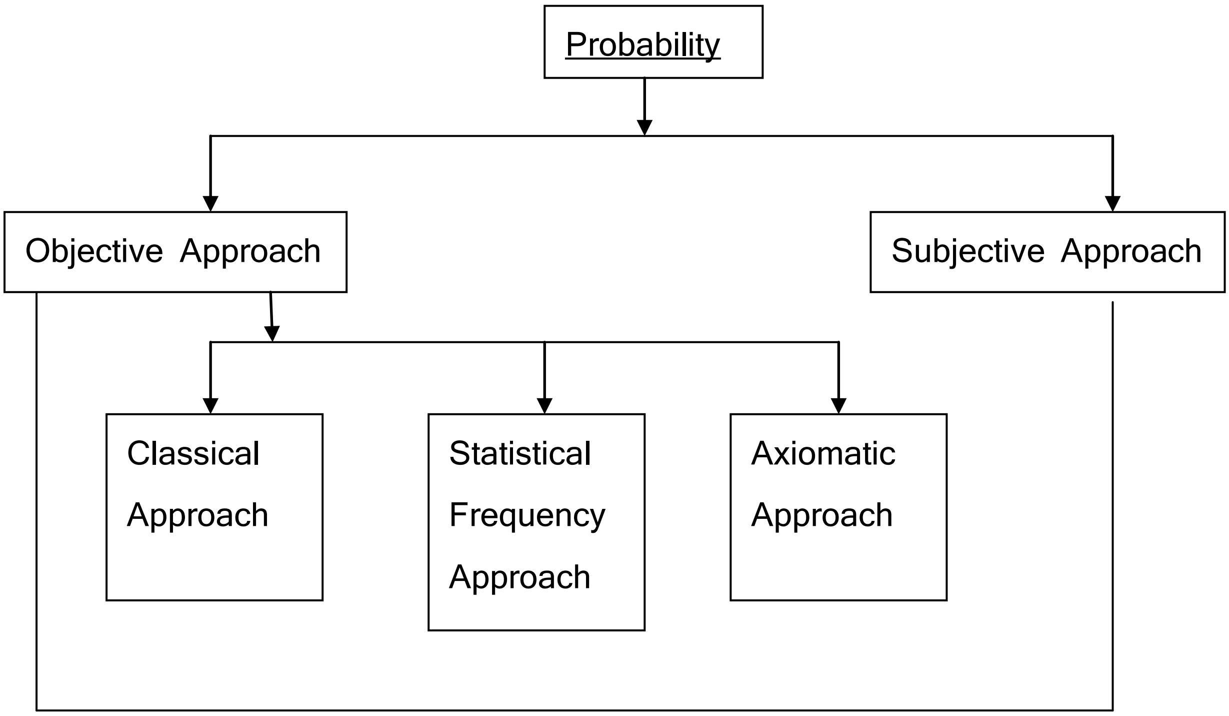
(5) Solve $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$

Given $\theta(0,t) = 0$, $\theta(x,0) = 2x$ for 0<x<π ,t=0

Unit – 3
Module – 2

Probability Distribution

Measurements of probability :



Review OF Probability

The Theory of Probability provides a means of getting an idea of the likelihood of occurrence of different events resulting from a random experiment in terms of quantitative measures ranging between zero and one. The probability is zero for an impossible event and one for an event which is certain to occur. The other degree of uncertain or the likelihood of occurrence of event are indicated by probabilities ranging between zero and one. In other words Probability is a concept which numerically measure the degree of uncertain and therefore of certain of the occurrence of events.

The theory of probability has its origin in the games of chance related to gambling with throwing a die, tossing a coin or drawing cards from a pack. Probability theory is now being applied in the analysis of social economic and business problems. **It has commercial application in the field of insurance where precise knowledge of risk to life or loss is required to enumerate premium.** In various activities of business, we face uncertainty and use probability theory for making management decisions. It is an essential tool in " Statistics Inference " and forms the basis of " Decision theory ". In fact, Statistics and Probability are very much interrelated and it's difficult to understand statistics without the knowledge of probability. The probability of Statistics is meaning use the bio – Statistical data.

For understanding of the concept probability theory, the following concepts must be ckoly grasped.

Terminology used in Probability:

Random Events: If in each trail of an experiment conducted under identical conditions the outcome is not unique but may be any of the possible outcome then such an experiment is called a Random Experiment.
For Example: If we toss a fair coin we may get a head or tail. Another experiment may be tossing a die in which three or six possible outcome or events i.e. the turning up of any of the six numbers 1, 2, 3, 4, 5 or 6. Further, the results of random experiment are called **Outcome Or Events.**

Equally Likely Events: Two or more events are said to be equally liked or equally possible if any of them cannot be expected to occur in preference to others. In other words events are called equally likely if the likelihood of the occurrence of every events is the same.
For Example: In tossing an unbiased coin the head and tail have an equal chance of turning up. Similarly, in throwing an unbiased die, all the possible outcomes 1, 2, 3, 4, 5, or 6 are equally likely.

Mutually Exclusive Events: Two events are called mutually exclusive when occurrence of one implies that the other cannot occur. In other words, events are called mutually exclusive if the occurrence of one includes the occurrence of the others.
For Example: In tossing a coin either head or tail occurs, i.e. the two events, Head and Tail, cannot occur simultaneously. Similarly, in throwing a die the occurrence of any number excludes the occurrence of the other and as such these six events are mutually exclusive.

Favorable And Unfavorable Events: The outcome is an experiment which are favorable to an event in which we are interested are called Favorable cases and all other outcomes are known as Unfavorable cases.
For Example: In throwing a die, to have the even numbers 2, 4 and 6 are favorable cases **.**

Independent Events: Two events may be independent, when the actual happening of one does not influence in any way the probability of the happening of the other.
Example: The event of getting head on first coin and the event of getting tail on the second coins in a simultaneously throw of two coins are independent.

Compound (or Joint) Events: When two or more events occur in composition with each other, the simultaneous occurrence is called a Compound Event.
Example: When a die is thrown, Getting a 5 or 6 is a compound events.
The Compound events may be further classified as :-

1. Independent Events.
2. Dependent Events.

Independent Events: If two or more events occur in such a way that the occurrence of one does not effect the other occurrence of another, they are said to be Independent Events.
For Example: If a coin is tossed twice, the results of the second throw would in no way be affected by the result of the first throw.

Dependent Events: If the occurrence of one event influences the occurrence of the other, then the second event is said to be Dependent on the first.
For Example: In the above example, if we do not replace the first ball drawn, this will change the composition of balls in the bag while making the second draw and therefore the events of drawing a red ball in the second draw will depend on event (First ball is red or white) occurring in first draw.

Definitions Of probability: -

We will discuss the following definition of probability:

1. Mathematical, Classical or a priori probability.
2. Statistical, empirical or a posteriori probability.
3. Axiomatic approach to probability.

Mathematical or Classical Probability: If consistent with the conditions of an experiment, then are exhaustive mutually exclusive and equally likely cases and of them are favorable to the occurrence of an event A, then the probability of happening of the event A, denoted as P(A) is :

$$P(A) = \frac{m}{n} = \frac{Number\ \ of\ favourable\ \ case}{Number\ \ of\ exaustive\ \ cases} \qquad \text{- (1)}$$

And the probability that the event A does not happen will be:

$$P\overline{(A)} = \frac{n-m}{n} = \frac{Number\ \ of\ cases\ unfavourable\ \ to\ the\ event\ A}{Exhaustive\ \ number\ \ of\ cases} \quad \text{- (2)}$$

Clearly,

$$P\overline{(A)} = 1 - \frac{m}{n} = 1 - P(A) \quad \text{- (3)}$$

Or $P(A) = 1 - P(\overline{A})$ - (4)

Or, $P(A) + P(\overline{A}) = 1$ - (5)

Thus, if we know the probability of an event A, then the probability of its complementary event, $\bar{A}$ is given by the formula (3), from the mathematical definition of probability.

Statistical or Empirical definition of probability: The classical definition of probability requires that **n** is finite and that all cases are equally likely. These are very restrictive condition and, as such, cannot all the situations. For overcoming such situations, the statistical is useful. According to this definition, if $(^{m}/_{n})$ is the relative frequency or frequency ratio od an event A comeeted with a random experiment, then the limiting value of the ratio $^{m}/_{n}$ as n increase infinitely is called the probability of the event A.
Symbolically,

$$P(A) = \lim_{n\to\infty} \left(\frac{m}{n}\right)$$

S.No	Classical Approach	Frequency Approach	Axiomatic Approach
1.	Suppose an experiment has a finite number (N) of "equal likely" outcomes P(A) =number of outcomes in $^{A}/_{N}$	Suppose that an experiment is conducted n times. Let n(A) denote the number of times the event A, occurs. Intuitively it suggests that P(A) can be approximated with $\frac{n(A)}{n}$	Satisfies number of axioms.
2	Probability of an event is calculated using counting techniques.	$\frac{n(A)}{n}$ will approach P(A) as no approach infinity.	Useful in developing the theory of probability.
3	Simulating Probabilities of event will further strengthen the understanding of this concept.	The conditions of experiment are not always static.	Doesn't tell how to compute the probability of an event.

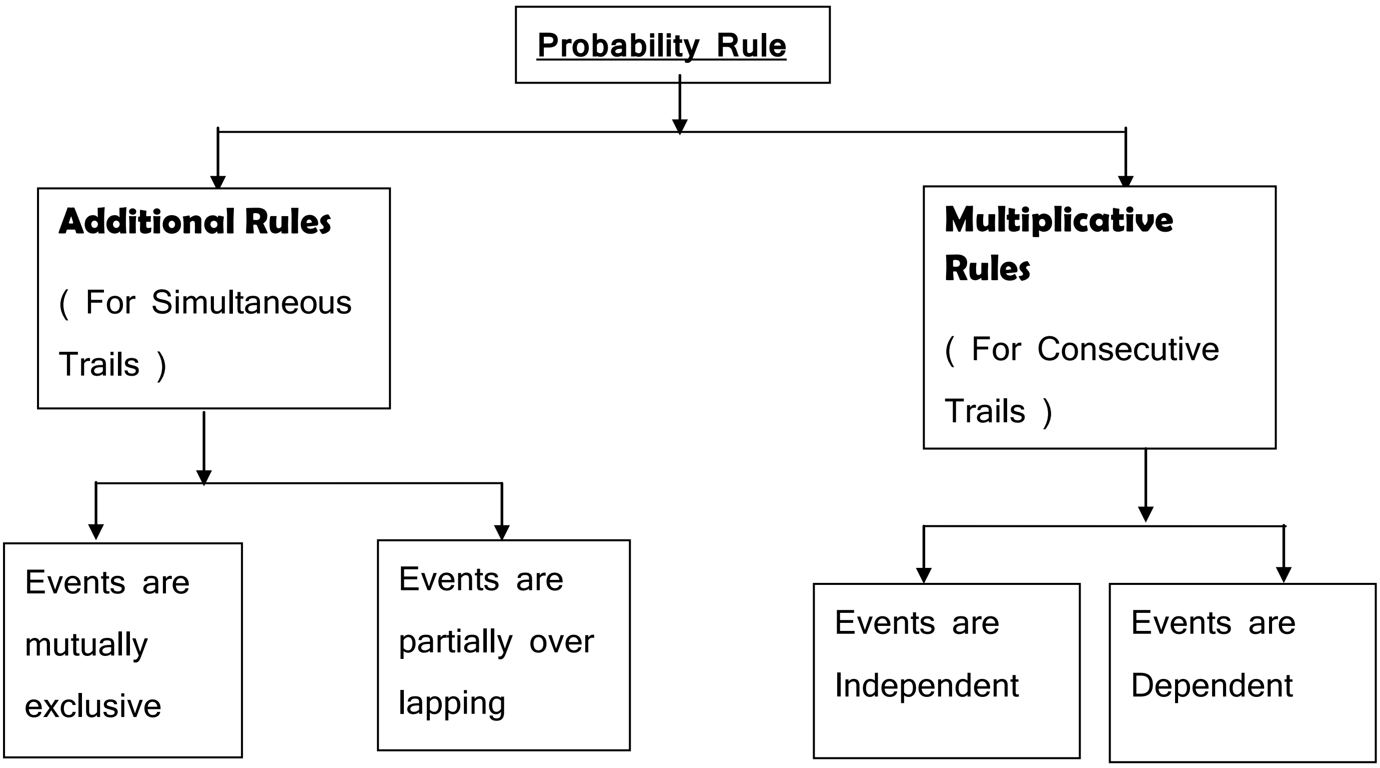
Axiomatic approach to the probability: For its proper understanding we need to study the concept of set and certain set operation which are discussed below:-

Theorems of probability:

There are two theorems of probability:

(1). The Addition Theorem.

(2). The multiplication Theorem.



Addition Theorem : If A and B are only two events associated with a random experiment, then,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Multiplication Theorem : If A and B are two events then,

$P(A \cap B) = P(B) \cdot P(A/B)$, Where $P(B) \neq 0$

Or, $P(A \cap B) = P(A) \cdot P(B/A)$, Where $P(A) \neq 0$

<u>Some basic concepts of Set Theory</u>:

Set A – A set is a collection of distinct and well defined objects. These objects forming the set are called the elements or members of the set. Distinct means that no two elements in a set are the same and well defined means that on the same and well defined means that on the basis of a rule it should be absolutely clear whether a particular object belongs to a particular set or not.,

Set are usually denoted by capital letters A, B X or Y. Where as small Letters a, b, x or y are used to indicate elements of a set.

For Example: The set A of possible outcomes when a die is tossed may be written as

A = { 1, 2, 3, 4, 5, 6 }

In set notations the symbol € means " is a element of " or "belongs to " and ¢ means " is not an element of " or does not belong to ".

For Example: If x is an element of set A and y is not then symbolically we write

x ¢ A and y € A

Similarly, For the set A = { 4, 5, 6, } , 4 € A but 7 ¢ A

<u>Subset</u>: A set A is called a subset of a set B if each element of set A also belongs to set B. Set A can be smaller or equal to set B.

<u>For Example</u>: B = { 1, 2, 3, 4, 5 }

The subset A = { 2, 3, 4 }

Then A is Subset of B.

A € B

<u>Equal Sets</u>: Two sets A and B are said to be Equal if all the elements of set A belongs to set B and all the elements of set B belongs to set A.

A = { 2, 4, 8 }

B = { 4, 2, 8 }

Then, A = B

Null (or Empty or void) Set: It is a set having no elements. It is denoted by { } or Ø.

Disjoint Set: Two sets A and B are said to be disjoint if there is no element common in them, this is, if there is no element which belongs to both set A and set B.

A = { 1, 2, 3, 4 }

B = { 6, 7, 8 }

Then A and B both are disjoint set.

Intersection of Sets: If A and B are two sets then their intersection is the set of those elements which are common to both the set A and B. It is denoted by A n B.

For Example:

If A = { 1, 2, 3, 4, 5 }

B = { 4, 5, 6, 7, 8 }

Then,

A n B = { 4, 5 }

Union of sets: The union of two sets A and B is set of elements that belongs to A or B or to both. Symbolically we write A U B for the union of A and B.

For Example:

A = { 1, 2, 3, 4, 5 }
B = { 4, 5, 6, 7, 8 }

Then,

A U B = { 1, 2, 3, 4, 5, 6, 7, 8 }

Complement of Set: Let U be any set (the universal set) and set A be its subset. Then the complement of set A in relation to U is that set B whose elements belongs to U but not to A. Complement of A is denoted by $\bar{A}$, A^c.

For Example:

U = { 1, 2, 3, 4, 5, 6, 7, 8, 9 }

And set A = { 6, 7, 8, 9 }

The complement of set A, A^c = { 1, 2, 3, 4, 5, }

Factorial Symbol: In the following rules we will observe that the products of consecutive integers are involved. We represent this product by a factorial symbol.

For Example: The product of (5 x 4 x 3 x 2 x 1) is written as 5! Or $L5$

In general, for any positive integer n, the product n(n-1) (n-2) - - - (3) (2) (1) is represented by the symbol Ln or n! Which is read as 'n factorial'

By definition $L1$ = $L0$ =1

Permutation and combinations: - 'The word permutation refers to the arrangement and the word combination refers to groups. These terms find their usage in the calculation of probability –

The number of permutation of n dissimilar things takes r at a time

$^nP_r = \frac{n!}{(n-r)!}$

The number of combination of n dissimilar things takes r at a time

$^nC_r = \frac{n!}{r!(n-r)!}$

Theoretical Probability Distributions :- Frequency distribution can be obtained in two ways via

i. By compiling actual frequencies through collection of data or by conducting experiments.
ii. By deriving the frequencies on the basis of some mathematical relationship.

Frequency distributed can be classified under two heads –

i. Observed frequency distribution.
ii. Theoretical (or expected) frequency distribution.

Observed frequency distribution are based on actual observation and experimentation. These are certain situation where we can drive expected values on the basis of some mathematical relationship. This distributions would be called a **Theoretical frequency distribution or a probability distribution.**

Thus, we can define theoretical or probability distribution as such distribution which are not obtained by actual observations or experiments but are mathematically deduced on certain assumptions.

Theoretical probability distribution are of two types –

i. Discrete probability distribution – Binomial, poission, geometric, hypergeometric, multinomial, etc.
ii. Continuous propability distribution – Normal, uniform, exponential, chi-square, Beta, t, F-distribution

We will however discuss here only there such distribution which are popular than others and are widely used –

i. Binomial distribution – Due to James Bernoulli (published in 1713)
ii. Poission distribution – Due to S.D. Poission (published in 1837)
iii. Normal distribution – Due to Demoivre (published close to the 18th century).

Binomial Distribution:– The binomial distribution in also known as the Bernoulli distribution in honour of the swiss mathematical Jacob Bernoulli (1654 – 1705) who derived it. Bernoulli distribution is a particular case of multinomial distribution and is of very great importance in research and problems connected with probability and sampling to begin with, we go back to our frequently used example of a fair coin. Assuming that the coin is tassed once, there can be two possibilities – either head (or success) or tail (or failure).

The sum of the probabilities is p + q where p is the probability of success and failure we may also say 1 and 0.

Experiment of this nature are called Bernoulli trail. Thus the Bernoulli trails have the following properties –

1. Each trail has only two possible outcomes 'success' or 'failure'.
2. The repeated trails are independent.

The probability of success in each trail remains constant.

<u>Binomial propability distribution</u>:– It has been developed to find the probability of r successes in n Bernoulli's trails, m thin regard, let us suppose that –

1. The experiment consists of n repeated trail.
2. In a trail, the occurrence of on event be considered as 'success; and non – occurrence as 'failure'. Let p be the probability of success and q = (1 – p) be the probability of failure in single trail.
3. Since r denote the number of successes in n independent repeated trails. Therefore r is a random variable which can take any of the values 0,1,2,3,- - - n.

With these notations, let us first consider the probability of r successes and (n – r) failure in a specific order. Here each success occurs and (n – r) failure in the specified order can be obtained by using multiplicative rule of probability as under –

p.p.p………..p q.q.q.q…….q

r times (n – r)times

$= P^r Q^{(n-r)}$

But we are interested in any r trails resulting in success and these r trails out of n can be chose n in ^nC_r mutually exclusive ways of ordering them. Thus on adding the probability of all ^nC_r.

Application of pronominal distributions – The following conditions are related to the binomial distribution –

1. Each trail result in two mutually disappoint outcome, called as success and failure.
2. The number of trail 'n' in finite.
3. The trails are independent to each other.
4. The probability of success p is constant for each trail.

Moment generating function of Binomial Distribution:

1. **<u>About mean</u>** :

$$M_{r-np}(t) = E\,[e^{t(r-np)}]$$
$$= e^{-npt}E(e^{tr})$$
$$= e^{-nptMr(t) =} e^{-npt}(q + pe^t)^n$$

$= (qe^{-pt} + pe^{t-pt})^n$

$= (qe^{-pt} + pe^{qt})^n$ $(q= 1 – p)$

2. **About Origin**:

$Mr(t) = E (e^{tr})$

$\sum_{r=0}^{n} etr\, {}^nc_r\, p^r q^{n-r}$

$\sum_{r=0}^{n} nCr\ (pe^t)^r\ q^{n-r}$

$= (q = pe^t)^n$

Cases or simply multiplying $p^r\ q^{(h-r)}$ by ${}^nC_r$, we get the general formula of computing the probability of r success in n.

Bemoulli trails as:

$P(x) = p^r q^{(n-r)}+p^r q^{(n-r)} + - - - + p^r q^{(n-r)}$ (${}^nC_r$ times)

- (1) $P(x=r) = {}^nC_r\ p^r q^{n-r}$

Where r= 0, 1, 2 ----- n

Now, by putting each value of the random variable r in equation (1), we get the binomial probability distribution as shown in the following table.

Number of success (x)	Probability function $P(r) = {}^nC_r p^r q^{n-r}$
0	${}^nC_0\ p^0 q^n = q^n$
1	${}^nC_1\ p^1 q^{n-1}$
2	${}^nC_2\ p^2 q^{n-2}$
3	${}^nC_3\ p^3 q^{n-3}$
.	.
r	${}^nC_r\ p^r q^{n-r}$
.	
n-1	${}^nC_{n-1}\ p^{n-1} q^{n(n-1)}$
n	${}^nC_n\ p^n q^0 = p^n$
Total	$(q + p)^n = 1$

Properties of the binomial probability Distribution-

1. N and p are the two parameters of the binomial distribution. As soon as the values of n and p are known, the binomial distribution is completely determined.
2. The mean of the binomial distribution is np.
3. If p = q = $1/2$, then the binomial distribution is a symmetrical distribution.
4. For p ≠ q, the binomial distribution is a skewed distribution.
5. Binomial distribution is a discrete probability distribution.

Mean and variance of binomial distribution

The Mean: The mean of binomial random variable X, denoted by M or E(X) is the theoretical expected number of successes in n trails.

$\mu = E(X) = \sum r.f(x)$

i.e. the mean of X is the sum of the products of the values that X can assume multiplies by their respective probabilities:-

$\mu = E(X) = \sum r.f(x)$

$= r\ {}^{n}C_{r}p^{r}q^{n-r}$

$= \sum_{r=o}^{n} \frac{n!}{x\ (n-r)!}\ p^{r}q^{n-r}$

$= \sum r \frac{n(n-1)!}{(r-1)!(n-r)!}\ p^{r-1}q^{n-r}$

$= \sum_{r=1}^{n} np \frac{n(n-1)!}{(r-1)!(n-r)!}\ p^{r-1}q^{n-r}$

$= np \sum {}^{n-1}C_{r-1}\ p^{n-1}\ q^{n-r}$

$= np\ (q + p)^{n-1}$ $\{(q + p)^{n-1}\}$

Hence, the mean of the binomial distribution is np

i.e. mean = np

The Variance: The variance of the binomial random variable X measure the variation of the binomial distribution and is given by-

$\sigma^2 = E(X^2) - \mu^2$

$= \sum_{r=0}^{n} r^2 f(x) - \mu^2$

Here, $\mu = np$

$\sum r^2 f(x) = \sum [r(r-1) + r]\ {}^{n}C_{r\ r}p^{2}q^{n-r}$

$= \sum r\ (r-1)\ {}^{n}C_{r}\ p^{r}q^{n-r} + \sum r\ {}^{n}C_{r}\ p^{2}q^{n-r}$

$= n(n-1)p^2\ (q + p)^{n-2} + np$

$= n(n-1)p^2 + np$ $(q + p^{n-2} = 1)$

Therefore,

$$\sigma^2 = n(n - 1)p^2 + np - (np)^2$$
$$= np [(n - 1)p + 1 - np]$$
$$= np (1 - q)$$

Variance (σ^2) = npq {Since p = q = 1}

Thus the standard demotion of the binomial distribution is $\sqrt{npq}$ and variance = npq

Example: The mean of a binomial distribution is 40 and standard deviation 6. Calculate n , p and q?

Solution: The mean of binomial distribution is given by np and standard deviation is given by $\sqrt{npq}$

Since, $\sqrt{npq}$ = 6 => npq = 36 and np = 40

Therefore,

$$40q = 36$$
$$q = \frac{36}{40} = 0.9$$

Then, p = 1 – q

P = 1 – 0.9

P = 0.1

Given,

$$np = 40$$
$$n(0.1) = 40$$
$$n = \frac{40}{0.1} = 400$$

p = 0.1

and q = 0.9 **(Answer**)

Example: The incidence of occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of sex workers 4 or more will contract disease ?

Solution: The probability of a worker who is suffering from the disease i.e.

$$p = 20\% = \frac{20}{100} = \frac{1}{5}$$

The probability of a worker who is not suffering from the disease,

i.e. $q = 1 - p = 1 - \frac{1}{5} = {}^{4}/_{5}$

The probability of 4 pr more i.e 4, 5 or 6 will contract disease is given by:

$P[\, x \geq 4 \,] = P(4) = P(5) + P(6)$ - (1)

By binomial distribution

$P(r) = {}^{n}C_{r}\, p^{r} q^{n-r}$

$P(r) = {}^{6}C_{r} \left(\frac{1}{5}\right)^{r} \left(\frac{4}{5}\right)^{6-r}$ - (2)

Using equation (2) in equation (1) find, P (4), P (5) and P(6).

Using (1)

$P[x \geq 4] = P(4) + P(5) + P(6)$ -(1)

By binomial distribution

$P(x) = {}^{n}C_{r}\, p^{r} q^{n-r}$

$P(x) = {}^{6}C_{r} \left(\frac{1}{5}\right)^{r} \left(\frac{4}{5}\right)^{6-r}$ -(2)

Using equation (2) in equation (1). Find P(4), P(5) and P (6)

Using (1)

$P[x \_ 4] = P(4) + P(5) + P(6)$

$= {}^{6}C_{4} \left(\frac{1}{5}\right)^{4} + {}^{6}C_{5} \left(\frac{1}{5}\right)^{5} \left(\frac{4}{5}\right)^{1} + {}^{6}C_{6} \left(\frac{1}{5}\right)^{6}$

$= 15 \left(\frac{1}{5}\right)^{4} \left(\frac{4}{5}\right)^{2} + 6\left(\frac{1}{5}\right)^{5} \left(\frac{4}{5}\right)^{1} + \left(\frac{1}{5}\right)^{6}$

$= \frac{15 X 16}{15625} + \frac{6 X 4}{15625} + \frac{1}{15625}$

$= \frac{1}{15625} [\, 240 + 24 + 1 \,] = \frac{265}{15625}$

$= 0.001696$ **(Answer)**

Example: Assume that on an average one telephone number out of fifteen is busy. What is the probability that if six randomly selected telephone number are called.

a) Not more than three will be busy ?

b) At least three of them will be busy ?

Solution:

p = probability that a telephone number is busy,

$p = \frac{1}{15}$

then, $q = 1 - p = 1 - \frac{1}{15} = \frac{14}{15}$ and $n = 6$

then Binomial distribution,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$P(r) = {}^6C_r \left(\frac{1}{15}\right)^r \left(\frac{14}{15}\right)^{6-r}$ - (1)

The probability that out of six ramdomly selected telephone numbers not more than three numbers are busy is given by

$P[x \le 3] = P(0) + P(1) + P(2) + P(3)$

Put r= 0, 1, 2, 3 in equation (1) then

Find the value of P(0), P(1), P(2), P(3)

$P[x \le 3] = {}^6C_0 \left(\frac{14}{15}\right)^6 + {}^6C_1\left(\frac{14}{15}\right)^5 \left(\frac{1}{15}\right)^1 + {}^6C_2\left(\frac{14}{15}\right)^4 \left(\frac{1}{15}\right)^2 + {}^6C_3\left(\frac{14}{15}\right)^3 \left(\frac{1}{15}\right)^3$

$= \left(\frac{1}{15}\right)^6 [\,(14)^6 + 6(14)^5 + 15(14)^4 + 20(14)^3\,]$

$= \frac{(14)^3}{(15)^6} [\,(14)^3 + 6(14)^2 + 15(14) + 20\,]$

$= \frac{2744}{(15)^6} [\,2744 + 1176 + 210 + 20\,]$

$= \frac{2744\ X\ 4150}{11390625} = 0.9997$

Probability that at least three telephone numbers are busy is given by:

P **[x >_ 3]** = P(3) + P(4) + P(5) + P(6)

Using equation (1) Put r = 3, 4, 5, 6

We get,

P**[x >= 3]** $= {}^6C_3 \left(\frac{14}{15}\right)^3 \left(\frac{1}{15}\right)^3 + {}^6C_4 \left(\frac{14}{15}\right)^2 \left(\frac{1}{15}\right)^4 + {}^6C_5 \left(\frac{14}{15}\right) \left(\frac{1}{15}\right)^4 + {}^6C_6 \left(\frac{1}{15}\right)^6$

= 0.0051 **(Answer)**

Example: If mean and variance of a binomial distribution are 4 and 2 respectively. Find the probability of:

I. Exactly 2 success
II. Less than 2 success
III. At least 2 success

Solution: The mean, np = 4 -(1)

Variance, npq = 2 -(2)

Diving equation (2) by (1), we get

$\frac{npq}{np} = \frac{2}{4}$, q = $\frac{1}{2}$

Then, p =1 – q

= 1 - $\frac{1}{2}$, p = $\frac{1}{2}$

Putting the value of p in equation (1) we get

np = 4

n($\frac{1}{2}$) = 4, n = 8

Hence Binomial distribution

$P(r) = {}^nC_r\, p^r q^{n-r}$

$P(r) = {}^8C_r(\frac{1}{2})^r\,(\frac{1}{2})^{8-r}$ -(3)

Probability of exactly 2 successes

P(2), Put , r = 2 in equation (3)

Then, $P(2) = {}^8C_2\,(\frac{1}{2})^2\,(\frac{1}{2})^6$

$= \frac{8 X 7}{2}$ X $\frac{1}{256}$

P(Less tham 2 successes) = P(0) + P(1)

Put r = 0 and 1 in equation (3), we get

= P(0) + P(1)

$= 8C0\,(\frac{1}{2})^0\,(\frac{1}{2})^{8-0} + {}^8C_1\,(\frac{1}{2})^1(\frac{1}{2})^3$

$= \frac{1}{256} + 8\frac{1}{2}(\frac{1}{2})^7$

$= \frac{9}{256}$ = 0.0351

iii) P (at Least 2 successes)

= P(2) + P(3) + P(4) + ----+ P(8)

= 1 – [P(0) + P(1)]

$= 1 - \frac{9}{256} = \frac{9247}{256}$ = 0.9648 **(Answer)**

Example: Out of 800families with 4 children each, how many families would be expected to have

i) 2 boys and 2 girls

ii) at least one boy

iii) no girl

iv) at most 2 girls ?

Assume equal probability for boys and girls.

Solution: Given probability for boys and girls are equal

p = probability of having a boy

$p = \frac{1}{2}$

q = probability of having a girl = $\frac{1}{2}$

n = 4, N = 800

Binomial Distribution

$P(r) = {}^{n}C_r\, p^r\, q^{n-r}$

$P(r) = {}^{4}C_r\, (\frac{1}{2})^r\, (\frac{1}{2})^{4-r}$ - (1)

The expected number of families having 2 boys and 2 girls

$= 800\ {}^{4}C_r\, (\frac{1}{2})^2\, (\frac{1}{2})^2$ $= 800 \times 6 \times \frac{1}{16} = 300$ family

ii) The expected number of families having at least one boy

= 800 [p(1) + p(2) + p(3) + p(4)]

$= 800\ [{}^{4}C_1\, (\frac{1}{2})^3\, (\frac{1}{2}) + {}^{4}C_r\, (\frac{1}{2})^2\, (\frac{1}{2})^2 + {}^{4}C_3\, (\frac{1}{2})\, (\frac{1}{2})^3 + {}^{4}C_4\, (\frac{1}{2})^4\,]$

$= 800 \text{ X } \frac{1}{16}\, (4 + 6 + 4 + 1)$

= 750 family

iii) The expected number of families having no girl i.e. having 4 boys

= 800 P(4)

$= 800\ {}^{4}C_4\, (\frac{1}{2})^4$ = 50 Family

iv)The expectes number of familyhaving at most two girl i.e. having at least 2 boys

= 800 [P(2) + P(3) +P(4)]

$= 800\ [\ {}^{4}C_2\, (\frac{1}{2})^2\, (\frac{1}{2})^2 + {}^{4}C_3\, (\frac{1}{2})\, (\frac{1}{2})^3 + {}^{4}C_4\, (\frac{1}{2})^4\,]$

$= 800 \text{ X } \frac{1}{16}\, (\ 6 + 4 + 1\ )$ = 550 Family **(Answer)**

Example: A binomial variable X satisfies the relation 9P(x=4) = P(x=2),
When n = 6, Find the value of the parameter p and P(X = 2)

Solution: We know that Binomial distribution $P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$ -(1)

$9 \, . \, P(X=4) = P(X=2)$

$9 \{ {}^{6}C_{4}\, p^{4} q^{6-4} \} = \{ {}^{6}C_{2}\, p^{2} q^{6-2} \}$

$9 \, {}^{6}C_{4}\, p^{2} q^{2} = {}^{6}C_{2}\, q^{4}$

$9 \; p^{2} = q^{2}$ $(q = 1 - p)$

$9 p^{2} = (1 - p)^{2}$

$9 p^{2} = 1 + p^{2} - 2p$

8p2 + 2p-1 = 0

$(4p - 1)(2p + 1) = 0$

$P = \frac{1}{4}$ $(p = -\frac{1}{2}$ = -ve negative)

Therefore, $q = 1 - p = 1 - \frac{1}{4}$

$= \frac{3}{4}$

Using equation (1)

$P(r) = {}^{4}C_{r} (\frac{1}{4})^{r} \frac{3}{4})^{4-r}$

Then Find,

$P(X=2) = {}^{4}C_{r} (\frac{1}{4})^{2} \frac{3}{4})^{2}$ **(Answer)**

Example: Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six ?

Solution: By the Binomial distribution

$P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$ -(1)

Let p = The chance of getting 5 or 6 with one dice

$= \frac{2}{6} = \frac{1}{3}$

Then $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

N = 6 , N = 729

Put all these values in equation (1)

$P(X=r) = {}^{6}C_{r} (\frac{1}{3})^{r} (\frac{1}{2})^{6-r}$ -(2)

Since dice are in sets of six and there are 729 sets

The expected number of times at least tree dice show 5 or 6

= N [P(3) + P(4) + P(5) + P(6)]

= 729 [$^6C_3\ (\frac{2}{3})^3\ (\frac{1}{3})^3 + {}^6C_4\ (\frac{2}{3})^2\ (\frac{1}{3})^4 + {}^6C_5\ (\frac{2}{3})(\frac{1}{3})^5 + {}^6C_3\ (\frac{1}{3})^6$]

= 293 **(Answer)**

Example: Assume that on the average one telephone number out of fifteen called between 2 P.M. and 3 P.M. on weak – days is busy. What is the probability that if 6 randomly selected telephone numbers are calles:

i) not more than three

ii) at least three of them will be busy.

Solution: Let p the probability of a telephone number being busy between 2P.M. & 3P.M. on week – days.

P = $\frac{1}{15}$

Then, q = 1 – p = 1 - $\frac{1}{15} = \frac{14}{15}$

n = 6

The probability that more than three will be busy: P(r ≤ 3)

= P(0) + P(1) + P(2) + P(3)

= $^6C_0\ (\frac{14}{15})^6\ (\frac{1}{15})^0 + {}^6C_1\ (\frac{14}{15})^5\ (\frac{1}{15})^1 + {}^6C_2\ (\frac{14}{15})^4\ (\frac{1}{15})^2 + {}^6C_3\ (\frac{14}{15})^3\ (\frac{1}{15})^3$

= $\frac{(14)^3}{(15)^6}$ [2744 + 1176 + 210 + 20]

= 0.9997

2). p (at least three of them will bw busy)

= p (r >= 3)

= P(3) + P(4) + P(5) + P(6)

= $^6C_3\ (\frac{14}{15})^3\ (\frac{1}{15})^3 + {}^6C_4\ (\frac{14}{15})^2(\frac{1}{15})^4 + {}^6C5(\frac{14}{15})^1(\frac{1}{15})^5 + {}^6C_6(\frac{1}{15})^6$

= 0.005

Example: Fit a binomial distribution to the following frequency data.

x	0	1	3	4
f	28	62	10	4

(UPTU-2004)

Solution: The data table is

x	**f**	**Fx**
0	28	0
1	62	62
3	10	30
4	4	16
	$\sum f$ **= 104**	$\sum fx$**= 108**

Mean = $\frac{\sum fx}{\sum f}$ = $\frac{108}{104}$ = 1.038

Mean, np = 1.038 q = 1 – p = 1 – 0.2595 = 0.7405

4p = 1.038 N = $\sum f$ = 104

P = 0.2595 Hence the binomial distribution = N(p+q) = $104(1)^n$

= 104

Exercise

1. What in meant by theoretical probability distributions.

2. Define binomial probability distribution and indicates its chief characteristics.

3. The sum and product of the mean and variance of a binomial distribution are $\frac{25}{3}$ and $\frac{50}{3}$ respectively, find the distribution.

4. In 800 families with 5 children each, how many families would be expected to have (i) 3 boys and 2 girls, (ii) 2 boys and 3 girls (iii) no girl (iv) at the most two girls. (Assume probability for boys and girls to be equal)

Answers: (i) 250 (ii) 250 (iii) 25 (iv) 400

5. The probability that a bulb product by a factory will fuse after a use of 150 days in 0.05. Find the probability that out of 5 such bulbs. (i) None (ii) Atmost one (iii) More than one (iv)such bulbs fuse after 150 days of use.

Answer: (i) $(\frac{19}{20})^5$ **(ii)** $\frac{6}{5}(\frac{19}{20})^4$ **(iii)1 -** $\frac{6}{5}(\frac{19}{20})^4$ **(iv)** $1-(\frac{19}{20})^5$

6. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random.

(i) 1 (ii) 0 (iii) At most 2, bolts will be defective.

Answer: (i) 0.4096 (ii) 0.4096 (iii) 0.9728

7. Four person in a group of 20 are graduates. If 4 persons are selected at random from 20. Find the probability that 10.

(i) All are graduate. (ii) at least one in graduate

(ii)

8. A product is 0.5% defective and is packed in cartons of 100. What percentage of its contain not more than 3 defective. (**Answer. 99.83%)**

9. If mean and variance of a binomial distribution are 4 and 2 respectively. Find the probability of (i) exactly two successes (ii) Less than two successes iii) more than 6 successes (iv) at least two successes.

Answer. (i) 0.1093 (ii) 0.0351 (iii) 0.0351 (iv) 0.9648

10. A carton contains 20 fuses, 5 of which are defective. Three fuses are chosen at probability that at most one defective fuse is found. (**Answer. 0.8437)**

11. An insurance salesman sells policies to 5 men, all of identical age in good health. According to the actuarial tables the probability that a man of this particular age will be alive 30 year hence is $\frac{2}{3}$. Find the probability that in 30 years.

1) All 5 men (B) at least 3 men (C) only 2 men (D) at least 1 man will be alive.

(Answer: (A) 0.1316 (B) 0.7901 (C) 0.1646 (D) 0.9958)

12. The overall percentage of failures in a certain examination is 20. If six candidates appear in the examination, What is the probability that at least five pass the examination. (**Answer. 0.65536)**

Poisson Distribution

Poission distribution was derived by S.D Poisson in 1837. This distribution is a discred probability distribution and in useful is such cases where the value of n is very large. In such cases the binomial distribution does not give appropriate theoretical frequencies. The poisson distribution in such situations has been found to be very appropriate. Poission distribution in a limiting from of binomial distribution as n moves towards infinity and p moves towards zero but np or mean remains constant and finite.

Conditions of Poisson Distribution :

1. The variable is discrete .
2. The numbers of trails i.e. n should be very large.
3. The probability of success i.e. p is very small.
4. The probability of success in each trail in constant.
5. Np is constant and finite.

Role of poisson distribution

The poisson distribution in used in practice in a wide variety of problem where there are infrequently occurring events with repeat to time area, volume or similar units . It can be used are given below.

1. It is used in quality control statistics to count the number of defects of an item.
2. In biology to count the number of bacteria.
3. In physics to count the number of particles emitted from a radioactive substance.
4. In insurance problems in count the number of casualties.
5. Number of traffic arrivals such as trucks at terminals, aeroplanes at airport, ships at decks and so forth.
6. In waiting – time problems to count the number of incoming telephone calls or incoming customers.

Poisson probability distribution

Poisson distribution is the discrete probability of a discrete random variable X which has no upper bound. Poisson distribution in a particular limiting form of the binomial distribution when p is very small and n is very large. Poisson distribution in defined for non – negative values of r as follows -

$$P(X = r) = \frac{e^{-m}.m^r}{L^r} \quad \text{For } r = 0, 1, 2....$$

Where m is the mean of distribution.

Thus, when n is large, p is small and np = m (constant), the limiting from of the binomial distribution is known as Poisson distribution.

Recurrence formula for Poisson Distribution

By Poisson distribution-

$$P(r) = \frac{e^{-m}.m^r}{r!} \quad -(1)$$

And, $P(r+1) = \frac{e^{-m}.m^{r+1}}{(r+1)!}$ --(2)

By dividing (2) by (1), we get,

$$\frac{P(r+1)}{P(r)} = \frac{e^{-m}.m^{r+1}}{(r+1)!} \text{ X } \frac{r!}{e^{-m}.m^r}$$

$$\frac{P(r+1)}{P(r)} = \frac{m}{r+1}$$

$$P(r+1) = \frac{m}{r+1} P(r)$$

Mean and variance of the poisson distribution

Mean: The mean of poisson distribution is given below:

$$\mu = \sum rf(r) = \sum r \frac{e^{-m}.m^r}{r!}$$

$$= 0 + me^{-m} + m^2e^{-m} + \frac{m^3e^{-m}}{2!} + \frac{m^4e^{-m}}{3!}$$

$$= me^{-m} [1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots\ldots]$$

$$= me^{-m}e^m$$

$$\mu = m$$

Thus, the mean of the Poisson distribution is mean (μ) = m

Variance: The variance of the poisson distribution is given by:

$\sigma^2 = \sum_{r=0}^{\infty} r^2$ P(r) – m^2

= $\sum_{r=0}^{\infty} r^2 \frac{e^{-m} m^2}{r!}$ - m^2

= $e^{-m} \sum_{r=1}^{\infty} \frac{r^2 m^2}{r!}$ – m

= e^{-m} [$\frac{1^2 m^1}{1!}$ + $\frac{2^2 m^2}{2!}$ + $\frac{3^2 m^3}{3!}$ + ….] – m^2

= me^{-m} [1+ $\frac{2m}{1!}$ + $\frac{3m^2}{2!}$ + $\frac{4m^3}{3!}$ + ….] – m^2

= me^{-m} [1 + $\frac{(1+1)m}{1!}$ + $\frac{(1+2)m^2}{2!}$ +$\frac{(1+3)m^3}{3!}$ + ….] –m^2

= me^{-m} [1 + $\frac{m}{1!}$ + $\frac{m^2}{2!}$ + $\frac{m^3}{3!}$ + ….] + ($\frac{m}{1!}$ + $\frac{m^2}{2!}$ + $\frac{m^3}{3!}$ + ….)] - m^2

= me^{-m} [e^m + m (1 + m + $\frac{m^2}{2!}$ + ….] – m^2

= me^{-m} [e^m + me^m] – m^2

= me^{-m} . e^m (1 + m) – m^2

= m (1 + m) – m^2

= m + m^2 – m^2

σ^2= m Variance = m

Thus the variance of the poisson distribution is also equal to m.

Moment Generating Function (MGF) of Poisson distribution

$M_x(t) = E(e^{tx})$

= $\sum_{x=0}^{\infty} e^{tx}$ P(X)

= $\sum_{x=0}^{\infty} \frac{e^{tx}, e^{-mx}}{x!}$ - m

= $\sum_{x=0}^{\infty} \frac{e^{-m}, (m\ e^t)^x}{x!}$ = $\sum_{x=0}^{\infty} \frac{e^{tx}, e^{-m}.\ m^x}{x!}$

= $\sum_{x=0}^{\infty} \frac{e^{-m}, (m\ e^t)^x}{x!}$

= $e^{-m} \sum_{x=0}^{\infty} \frac{(m\ e^t)^x}{x!}$

= $e^{-m} e^{met}$

= $M_x(t) = e^{m(et-1)}$

Example: A manufacturer knows that the razor blade he makes contains on an average 0.5% of defective. He packs them in a packets of 5, What is probability that a packet at random will contain 3 or more faulty blades?

Solution: Given

Let p = 0.5% = 0.005

N = 5

Then, mean m = np = 5 x 0.005

= 0.025

We know that Poisson probability diatribution is

$$P(X = r) = \frac{e^{-m} m^2}{r!}$$

$$P(X = r) = \frac{e^{-0.025} \; (0.025)^2}{r!} \qquad -(1)$$

P(3 or more faulty blades)

= P (x ≥ 3)

= P (3) + P (4)+P (5)

Put r = 3, 4, 5 in equation (1) we get

$$= \frac{-0.025 \; (0.025)^3}{3!} + \frac{e^{-0.025} \; 90.025)^4}{4!} + \frac{e^{-0.025} \; (0.025)^5}{5!}$$

= 2.555 X 10^{-6} **(Answer)**

Example: The distribution of the number of road accidents per day in city is Poisson with mean and find the number of days out of 100 days when there will be:

(i) No accident

(ii) At least 2 accidents

(iii) At least 3 accidents

(iv) Between 2 and 3 accidents

Solution: Given

Mean m = 4

By, Poisson distribution

$$P(r) = \frac{e^{-m} . m^r}{Lr}$$

$P(r) = \frac{e^{-4}.4^r}{r!}$ -(1)

(i) P(no accidehnt) = p(0)

Put r = 0 in equation (1)

$= \frac{e^{-4}.4^0}{0!} = e^{-4} = 0.0183$

Required number of the days = 100 x 0.0183= 1.83= 2 Days.

(ii) P(at least 2 accident)

= p (x ≥ 2) = P(2) + P(3) + P(4) . . . . .

= 1 – [P(0) + P(1)]

$= 1 - [\frac{e^{-4}.4^0}{0!} + \frac{e^{-4}.4^1}{1!}] = 0.9084$

Requires number of days = 100 x 0.9084 = 90.84 = 91 days

(iii) P (at most 3 accidents)

= p(0) + p(1) + p(2) +p(3)

Put r = 0, 1, 2, 3, in equation (1), we get

$= \frac{e^{-4}.4^0}{0!} + \frac{e^{-4}.4^1}{1!} + \frac{e^{-4}.4^2}{2!} + \frac{e^{-4}.4^3}{3!}$

= e-4 [1 + 4 + 8 + $\frac{32}{2}$]

= 0.4331

Required number of days = 100 x 0.4331

= 433.1

= 433 days

(iv) P (between 2 & 3 accidents)

= p (2 < r < 5)

= p (3) + p(4)

$= \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!} = 0.0183 + \frac{64}{4}$

= 0.3907

Required number of days = 100 x 0.3907 = 39 days. **(Answer)**

Example: If X is a Poisson variante such that

P (X = 2) = 9 . P (X = 4) + 90 . P (X = 6)

Then find mean m of X

Solution: We know that if X is a Poisson variante then

$$P (X = r) = \frac{e^{-m}.m^r}{r!} , r = 0 , 1, 2, \ldots \qquad -(1)$$

Then condition-

P (X =2) = 9 . P (X = 4) + 90 . P (X = 6)

P (2) = 9 . P (4) + 90 . P (6)

Using (1) …

$$\frac{e^{-m}.m^2}{2!} = 9 . \frac{e^{-m}.m^4}{4!} + 90 . \frac{e^{-m}.m^6}{6!}$$

$= m^4 + 3m^2 -4 = 0 \qquad -(2)$

Solving equation (2) for m^2 , we get

$m^2 = \frac{-3 \pm\sqrt{9+16}}{2}$

$m^2 = \frac{-3 \pm 5}{2}$

Since, m > 0, we get $m^2 = 1$

m = 1

Hence, Mean = variance = m = 1 **(Answer)**

Exapmle: Suppose that a manufactured product has on the average 3 defeats per unit of product inspected. Using Poisson distribution . Calculate the probability of finding a product :(i) Without any defect

(iii) with 3 defects

(iii) with 4 defects

(given , e^{-3} = 0.0495)

Solution: Given Mean m = average number of defects.

M = 3

Now poisson distribution is

$$P (X =R) = \frac{e^{-m}.m^r}{r!} \; r = 0, 1, 2, \ldots. \quad -(1)$$

(i) P (without any defect)

= p (0) , Put r = 0 in equation (1)

$= \frac{e^{-m}.0^r}{0!} = e^{-3} = 0.0495$

(ii) P (with 3 defects) = p (r = 3)

$$\frac{e^{-3}.3^3}{3!} = 0.22275$$

(iii) P (with 4 defects) = p(4)

$= \frac{e^{-3}.3^4}{4!}$ = 0.1670625 **(Answer)**

Example: If a random variable X follows Poisson distribution such that P(X-1) = P(X-2), then find :

(1) the mean of the distribution

(2) P (X = 0)

Solution: The poisson distribution is:

$$P(r) = \frac{e^{-m}.m^r}{r!} \quad -(1) \quad r = 0, 1, 2, ...$$

Since, P (X = 1) = P (X = 2)

$$\frac{e^{-m}.m^1}{1!} = \frac{e^{-m}.m^2}{2!}$$

(i) i.e. the mean of the distribution , m= 2

(ii) On putting m = 2 in equation (1), we get

$$P(r) = \frac{e^{-m}.m^2}{r!} = \frac{e^{-2}.2^r}{2!}$$

Then, $P(0) = \frac{e^{-2}.2^0}{0!} = e^{-2}$

= 0.1353 **(Answer)**

Example: A skilled typist on routine work, kept a record of mistake made per day during 300 working days.

Mistake per day (x)	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Number of days (f)	143	90	42	12	9	3	1

Fit the Poisson distribution to the above date and hence calculate the theoretical frequencies.

Solution: The mean number of mistakes

$$m = \frac{\sum fx}{\sum f} = \frac{\sum fx}{N} = \frac{267}{300} = 0.89$$

New frequency f(x) for x = 0, 1, 2, 3, 4, 5, 6. Are

F(0) = N . P(0) = 300{ $\frac{e^{-0.89}\ (0.89)^0}{0!}$

= 123.3 = 123

f(1) = N.P(1) = 109.5 = 110

f(2) = N.P(2) = 48.9 = 49

f(3) = N.P(3) = 14.4 = 14

f(4) = N.P(4) = 3.3 = 3

f(5) = N.P(5) = 0.6 = 1

f(6) = N.P(9) = 0.09 = 0

Hence,

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Extended Frequency	123	110	49	14	3	1	0

(Answer)

Example: A student has obtained the following answers to certain problems in statistics.

(i) Mean = 3 and variance = 4 for a Binomial distribution

(ii) Mean = 5 and variance = 6 for a Poisson distribution

Discuss the criticize the answer obtained by the student.

Solution: We know that the variance of binomial distribution is always less than its mean. In the given statement.

(i) Variance = 4 > mean = 3. Thus the answer obtained by the student is innocent further.

Mean = np = 3 and variance = npq = 4

Then $\frac{npq}{np} = \frac{4}{3}$ = 1.33 > 1

Since q is the probability of failure which cannot exceed 1, the answer obtained by the student is incorrect.

(ii) Variance = 6 > mean = 5

Here also,

np = 5 , npq =6

q = 1.2 > 1

Since q is the probability of failure which cannot exceed 1. The answer obtained by the student is incorrect. **(Answer)**

Exercise

1. Define Poisson distribution and state the condition under which this distribution is used.

2. In a certain manufacturing process 5% of the tools produced turn out to be defective. Fins the probability that in a sample of 40 tools, utmost 2 will be defective. [Given that e^{-2} = 0.135] **(Answer: 0.675)**

3. Using Poisson distribution, find the probability that the ace of shades will be drawn from a peak of well-shuffled cards at least once in 1.4 consecutive trails. (Given e^{-2} = 0.135)

4. In a certain factory producing cycle tyres there is a small chance of 1 in 500 tyres to be defective. The types are supplied in lots of 10. Using poisson distribution. Calculate the approximate number of lots containing no defective, one defective and two defective tyres respectively, in a consignment of 10,000 lots. **(Answer: 9802 lots , 196 lots, 2 lots)**

5. If the variance of the Poisson distribution on is 2. Find the probabilities for r = 1, 2, 3, 4. From the recurrence relation of the Poisson distribution, Also find P(r = 4). **(Answer: 0.1431)**

6. Fit a Poisson distribution to the following data and calculate theoretical frequencies:

Death	**0**	**1**	**2**	**3**	**4**
Frequencies	**122**	**60**	**15**	**2**	**1**

(Answer: 200)

7. Suppose that X has a Poisson distribution id
   P (x = 2) = $\frac{2}{3}$ P (X = 3)
   (i) P (X = 0) and (ii) P (X = 3)

8. A manufacturer of cotter pins known that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantee that not more than 10 pins will be defective. What is the approximate probability that a box fail to meet the guarantee quality (e^{-5} = 0.006738) **(Answer: 0.0136875)**

9. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the umber of days in a year on which (i) car is not used, (ii) the number of days in a year some demand is refused.

(Answer: (i) 81 days (ii) 70 days.)

10. Suppose that the number of telephone calls coming into the switch board between 10a.m. and 12 noon of a company is 2.5. Fins the probability that during one particular minute tere will be: (i) 0 phone (ii) 1 phone (iii) 2 phone calls **(Answer: (i) 0.0821 (ii) 0.2052 (iii) 0.2565)**

Normal distribution (or Gaussion Distribution)

The normal distribution (or Gaussion distribution) is a continuous, probability distribution that describes data the cluster around a mean. The graph of the associated probability density functions is bell-shaped, with a peak at the mean, and is known as the Gaussion function or bell curve.

The normal distribution can be used to describe at least approximately, any variable this tends to cluster around the mean.

The probability density function for a normal distribution is given by the formula:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ \frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\}$$

(For $-\infty \leq x \leq \infty$)

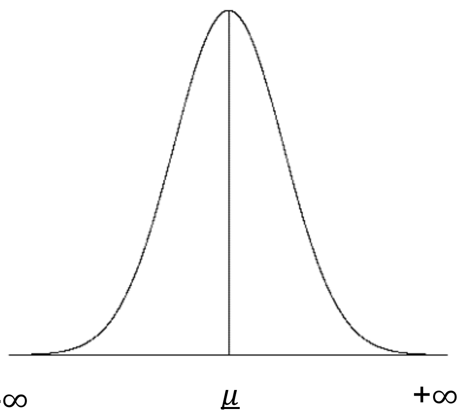
Where μ is the mean, 6 is the standard deviation (a measure of the ' width ' of the bell) and exp denotes the exponential functions. For mean of O and a standard deviation of 1. This formula is: $P(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Which isi known as the standard normal distribution.

μ and 6 are also known as the two parameters of the normal distribution. Once the values of μ and 6 are known the shape of the equation of the normal distribution is completely determined.

(1). De Moivre made the discovery of the normal distribution in 1733. Normal distribution is a continuous distribution. It is derived as the limiting from of the Binomial distribution for large values of n_1p and q are not very small.



The Normal Curve

Properties of the Normal distribution (or curve)

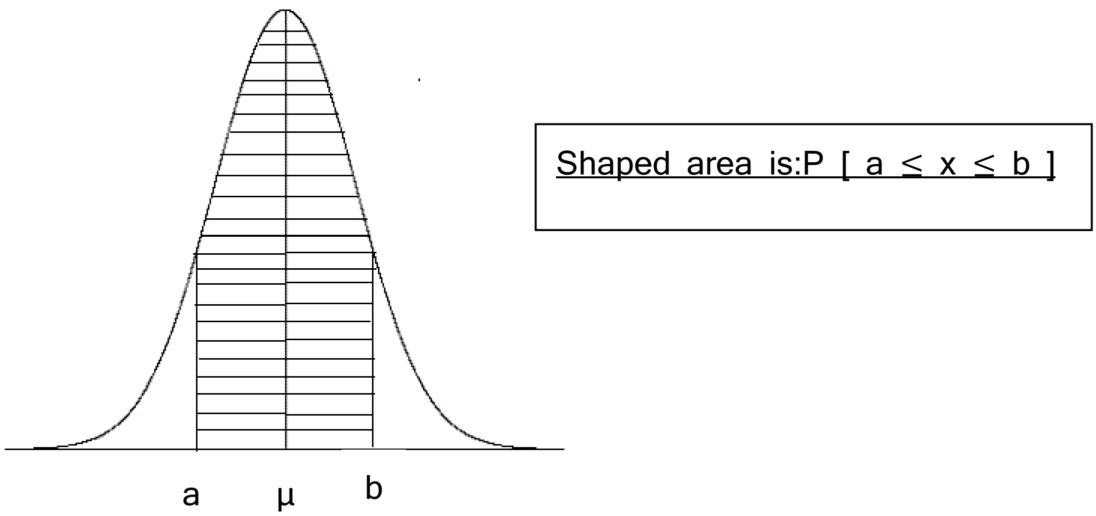
The properties of normal distribution or normal curve are given as:

1) The curve is symmetrical about the vertical axis through the mean that is if we fold the curve along with vertical axis, the two halves of the curve would coincide.
2) As the result of symmetry, the mean, medium and mode of the distribution are identical, Mean = Medium = Mode.
3) Since there is only one maximum point in the curve, the normal curve is unmade i.e. It has only one mode.
4) The total area under the normal curve and above the horizontal axis is unity which is essential for a probability distribution or a curve.
5) The mean deviation = $4/5$ Standard deviation.

Area Under the normal curves

The area of probability under a normal probability distribution or curve bounded by two parameter x = a and x = b is written as
P [a ≤ x ≤ b].

This is the probability that a normally distributed variable x lies between two specified value a and b and can be represented by the shaded area which is given below:



The area or probability under a normal curve depends on its parameter μ and **6.** Thus, the area under the normal curve will change with the value of μ and 6

It is possible to transform any normal variable X to standard normal variable Z by using the transformation:

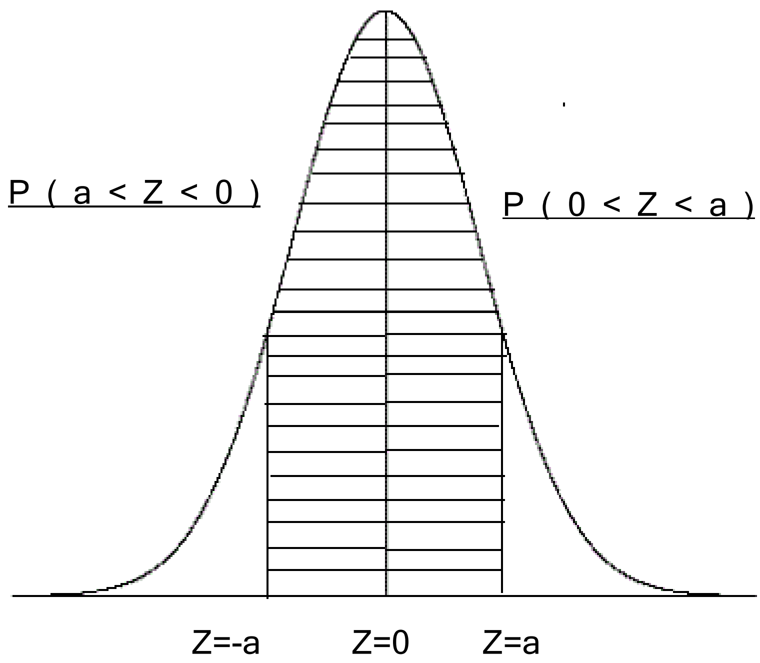
$$Z = \frac{X-\mu}{6}$$

How to use Table: The following important points should kept in mind while computing area under standard normal curve-

1) The total area under the standard normal curve is 1.
2) The mean of distribution is zero thus the negative and positive values of Z will lie on the left and right of mean respectively.

3) The ordinate at mean i.e. at Z = 0 divides the area under the standard normal curves into two equal parts. Thus, the area on the right and left of the ordinate at Z = 0 is 0.5.

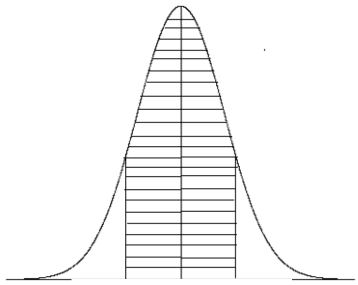
$P[-\infty \leq Z \leq 0] = P[0 \leq Z \leq \infty] = 0.5$



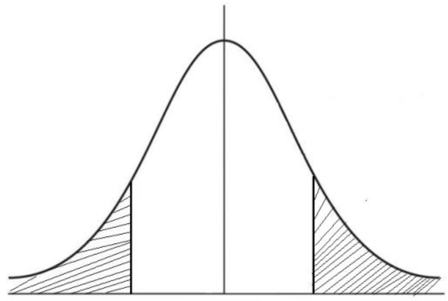
Example: If Z is a standard normal variable. Find the probabilities or areas.

I. P (-2 < Z < 1)
II. P (- ∞ < Z < -2)
III. P (2 < Z < ∞)

Solution: P (-2 < Z < 1) = P (-2 < Z < 0) + P (0 < Z < 1)
=P (0 < Z < 2) + P (0 < Z < 1)
= 0.4772 + 0.3413
= 0.8185



(ii). P (- ∞ < Z -2) = P (2 < Z < ∞) (From Symmentry)
= 0.5 – P (0 < Z < 2)
= 0.5 – 0.4772
= 0.0228



(iii) P (2 < Z < ∞) = 0.0228 **(Answer)**

Question : A sample of 100 day battery cells tested to find the Length of life produced the following results:

$\bar{x}$ = 12 Hours, σ = 3 hours

Assuming the data to be normally distributed, what percentage of battery cells are :

(i) More than 15 hours.
(ii) Less than 6 hours.
(iii) Between 10 and 18 hours. (AKTU = 2018)

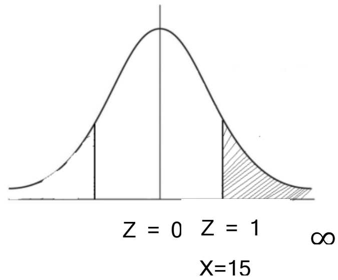
Solution: Here given, mean $\bar{x}$ = 12 hours

And standard deviation σ = 3 hours

Let x denotes the length of life of day be heavy cells

$$Z = \frac{x-\mu}{\sigma} = \frac{x-12}{3} \qquad - (1)$$

(i) P (more than 15 hours)



Put x = 15 in equation (1)

$Z = \frac{15-12}{3} = 1$

= P (x > 15) = P (Z > 1)

= P (0 < Z < ∞) – p (0 < Z < 1)

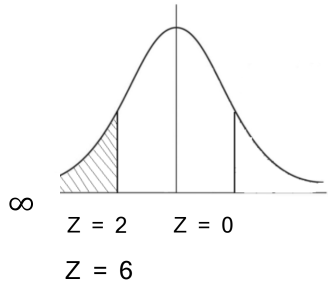
= 0.5 – 0.3413 = 0.1587

= 15.87 %

(ii) P (Less than 6 hours)

Put x = 6 in equation (1)

We get,



$$Z = \frac{6-12}{3} = 2$$

P (0 < Z < ∞) – P (0 < Z < 1)

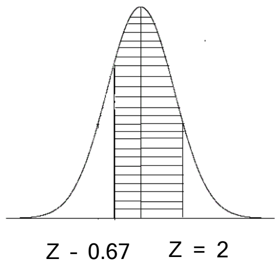
= 0.58 – 0.4772 = 0.028

= 2.28 %

(iii) P (between 10 and 18 hours)

Put x = 10 and also put x = 18 in equation (1)

$Z1 = \frac{10-12}{3} = -0.67 \quad Z2 = \frac{18-12}{3} = 2$



= P (10 < x < 18)

= P (-0.67 < Z < 2)

= P (-0.67 < Z < 0) + P (0 < Z < 2)

= 0.2485 + 0.4772

= 0.7257

= 72.57% **(Answer)**

Example: An Aptitude test was conducted on 900 employes of the metro tyres limited in which the mean score was found to be 50 with S.D of 20. On the basis of this information, you are required to answer the following questions:

a) What was the number of employees whose mean score was less than 30.
b) What was the number if employees whose mean score exceeded 70.
c) What was the number of employees whose mean score was between 30 and 70.

$\frac{x-4}{6}$	0.25	0.50	0.70	1.00	1.25	1.50
Area	0.0987	0.1915	0.2734	0.3413	0.3944	0.4332

Solution: Given the data,

Mean $\mu = 50$

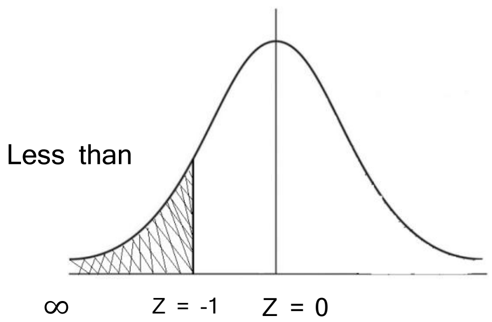
S. D. $\sigma = 20$

We know that, $Z = \frac{x-\mu}{\sigma} = \frac{x-50}{20}$ -(1)

(i) P (less than 30)

Put x = 30 in equation (1) we get,

$Z = \frac{30-50}{20} = -1$



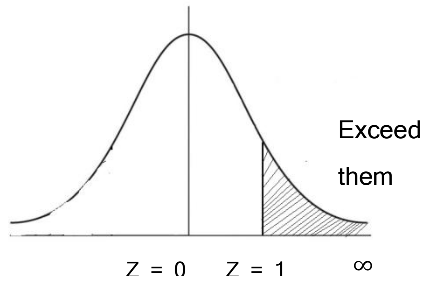
= P (X < 30)
= P (Z < -1)
= P (Z > 1) (From Frequency)
= P (- ∞ < Z < 0) – P (-1 < Z < 0)
= P (0 < Z < ∞) – P (0 < Z < 1)
= 0.5 – 0.3413 = 0.1587

Therefore, The number of employees whose mean score was less than 30
= 900 x 0.1587 = 143 (Approx)

(ii) P (mean score exceeded 70)

Put x = 70 in equation (1) we get

$Z = \frac{70-50}{20} = 1$



Therefore, P (x > 70)
= P (Z > 1)
= P (0 < Z < ∞) – P (0 < Z < 1)
= 0.5 – 0.3413 = 0.1587

Therefore, the number of employees whose mean score exceeded 70 is = 900 x 0.1587

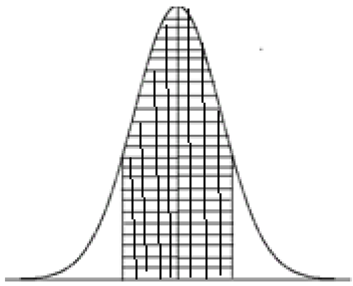
= 143 (Approx)

(iii) P (score between 30 and 70)

Put x = 30 and also put x = 70 in equation (1) we get-

$Z_1 = \frac{30-50}{20}$ $Z_2 = \frac{70-50}{2}$

$Z_1 = -1$ $Z_2 = 1$



= P (30 < x < 70)

= P (-1 < Z < 1)

= P (-1 < Z < 0) + P (0 < Z < 1)

= 0.3413 + 0.3413 = 0. 6826

Therefore, The number of employees whose mean scores are between 30 and 70 are = 900 x 0.6826

= 614 **(Answer)**

Example: In a sample of 1000 cases the mean of a certain test is 14 and S.D. is 2.5. Assuming the distribution on to be normal. Find:

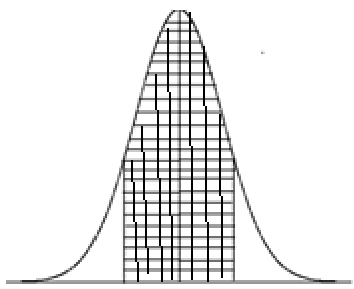
a) How many students score between 12 and 15.
b) How many score above 18
c) How many score below 8.

Solution: Given mean μ = 14

S.D. σ = 9.5

We know that,

$$Z \frac{x-\mu}{\sigma} = \frac{x-14}{2.5} \quad -(1)$$



P (between 12 and 15)

Put x = 12 and also put x = 15 in equation 1.

$Z_1 = \frac{12-14}{2.5}$ $Z_2 = \frac{15-14}{2.5}$

$Z_{1 = -0.8}$ $Z_{2 = 0.4}$

= P (12 < x < 15)

= P (-0.8 < Z ≤ 0.4)

= P (0 < Z ≤ 0.8) + P (0 < Z ≤ 0.4) (From symmetry)

= 0.2881 + 0.1554

= 0.4435

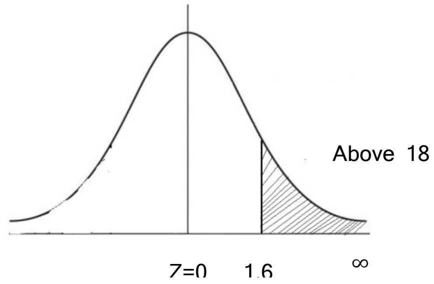
∴ Required number of students = 1000 x 0.4435

= 444 (Approx)

(b) P (score above 18)

Put x = 18 in equation (1) we get,

$Z = \frac{18-14}{2.5} = 1.6$



= P (x > 18)

= P (Z > 1.6)

= P (0 < Z ≤ ∞) – P (0 < Z ≤ 1.6)

= 0.5 – 0.4452 = 0.0548

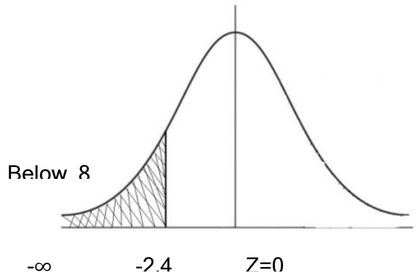
Required number of students = 1000 x 0.0548

= 55 (Approx)

(c) P (score below 8)

Put x = 8 in equation (1) , we get

$Z = \frac{8-14}{2.5} = -2.4$



P (x < 8)

= P (Z < -2.4)

= P (- ∞ < Z < 0)

= P (0 < Z ≤ ∞) – P (0 < Z ≤ 2.4) (From Symmetry)

= 0.5 – 0.4918

= 0.0082

Required number os students = 1000 x 0.0082

= 8.2

= 8 **(Answer)**

Example : In a normal distribution 31% os the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution.

(AKTU – 2018,19)

Solution: Let μ and 6 be the mean and S.D. respectively-

We know that- $Z = \frac{X - \mu}{\sigma}$ -(1)

Let $Z_1 = \frac{x1 - \mu}{\sigma}$ (2)

$Z_2 = \frac{x2 - \mu}{\sigma}$ (3)

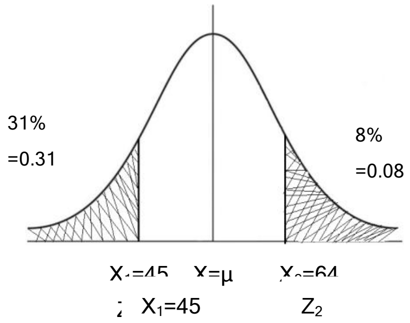
Now $P(Z_1 \leq Z \leq Z_2) = P(0 \leq Z \leq Z_1)$

$= 0.19$

$\Rightarrow Z_1 = -0.5 \; (Z_1 < 0)$

And, $P(0 < Z < Z_2) = 0.42$

$Z_2 = 1.4$



From equation (2) and (3), we get

$-0.5 = \frac{45 - \mu}{\sigma}$ (4)

$1.4 = \frac{64 - \mu}{\sigma}$ (5)

Solving equation (4) and (5), we get

Mean $\mu = 50$ and S.D. $(\sigma) = 10$ **(Answer)**

Relation Between Distributions:

Binomial and Poisson:

If n us larger, p is small and np = m (a positive finite constant 0, then binomial distribution tends to Poisson distribution.

Binomial and Normal:

If n is large neither p nor q is small, then the discrete binomial distribution tends to the continuous normal distribution with mean np and S.D. $\sqrt{npq}$.

Poisson and Normal:

The poisson distribution also trends to normal distribution when its parameter is increase indefinitely.

Exercise:

1) What is meant by theoretical frequency distribution. Discuss the salient features of the binomial, poisson and normal distributions?

2) Describe the Chief Characteristics of the normal curve. Why is this curve given a central place in statistics?

3) In an intelligence test administered to 1000 children, the average score is 42 and standard denatim 24.
   (i) Fins the number of children exceeding the score 60.
   (ii) Fins the number of children with score lying between 20 and 40 (Assume the normal distribution)

   (Answer: (i) 227 (ii) 289)

4) The standard devistion of a certain group od 1000 high school grades was 11% and the mean grade 78 %. Assuming the distribution to be normal . Find:
   (i) How many grades were above 90%.
   (ii) What was the highest grade of the lowest 10.

   (Answer: (i) 138 (ii) 53)

5) In a normal distribution, 7% of the items are under 35 and 89% are under 63. What are mean and standard deviation of the distribution?

   (Answer: mean = 50.3, S. D. = 10.33)

6) Determine the minimum marks of a student must get in order to receive on A grade if the top 10% of the students are awarded A grades in an examination where the mean mark is 75 and S.D. is 9 ?

   (Answer: 84)

7) The life of army shoes is normally distributed with mean 8 months and S.D. 2 months. If 500 pairs are issued how many pairs would be expected to need replacement after 12 months?

   (Answer: 4886)

8) Suppose the weight W of 600 male students are normally distributed with mean μ = 70 kg and standard deviation 6 = 5kg. Find number of students with weight:
   (i) Between 69 and 74kg
   (ii) More than 76 kg

(Answer: (i) 220 (ii) 69)

9) The daily wages of 1000 workers are distributed around a mean of Rs. 140 and with a standard deviation of Rs. 10. Estimate the number of workers whose daily wages will be:
   (i) Between Rs. 140 and Rs. 144
   (ii) Less than Rs. 126
   (iii) More than Rs. 160

(Answer: (i) 155 (ii) 81 (iii) 23) **(GBTU – 2012)**

10) The marks X obtained in mathematics by 1000 students are normally distributed with mean 78% and standard deviation 11%. Determine:
   (i) How many students got marks above 90%.
   (ii) What was the highest marls obtained by the lowest 10% of students.
   (iii) Within what limit did the middle 90% of the students lie?

(Answer: (i) 138 (ii) 63.4% (iii) Between 60 and 96)

11) A large number of measurement is normally distributed with a a mean 65.5" and S.D. of 6.2". Find the percentage of measurement that fall between 54.8" and 65.8". **(Answer: 66.01%)**

12) In a male population of 1000, the mean height is 68.16 inches and standard deviation is 3.2 inches. How many man may be more than 6 feet (72 inches)?
   P (Z = 1.15) [Ø (1.15) = 0.8749,
   P (z = 1.2) Ø (1.2) = 0.8849
   P (Z = Ø (1.25) = 0.8944]
   Where the argument is the standard normal variable.

(Answer: 115)

13) In a regiment of 1000, the mean height of the soldiers in 68.12 units and the standard deviation is 3.374 units. Assuming a normal distribution, how many soldiers could be expected to be more than 72 units. It is given that:

P (Z = 1.00) = 0.3413, P (Z = 1.15) =0.3749

P (z = 1.25) = 0.3944, Where Z is the Standard normal Variable.

(Answer: -125)

14) The mean yield per plot of a crop is 17kg and S.D. is 3 kg. If distribution of yield per plot is normal. Find the percentage of plots given yields:

(i) Between 15.5 kg and 20 kg.

(ii) More than 20 kg.

(Answer: (i) 53.28% (ii) 15.87%) (UPTU/ MBA- 2015)

15) A normal variable x, has mean 1 and variance 4. Find the probability that $x \geq 3$.

(Answer : 0.1587)

Module-III

Numerical Techniques

- Zeros of transcendental and polynomial equation
- Bisection method, Regular flare method Newton Ranbson method
- Rate of convergence of above methods

Interpolation

- Finite difference
- Newton's forward and backward interpolation
- Lagrange and Newton difference formula for unequal internals

Introduction

In scientific and engineering work, a wide variety of problems can be formulated into equations of the form
$f(x) = 0$

where x and f(x) may be real, complex or vector quantities.

The solutions process often involves finding the value of x that would satisfy the above equation. These values are called roots of the equation. Since the function f(x) becomes zoro at these values, they are also known as the zero of the function (x) Equation f(x) = 0 may belongs to one of the following types of equations:-

1. Algebraic Equations.
2. Polynomial Equations
3. Transcendental Equations

Algebraic Equations

An equation of type y = f (x) is said to be algebraic if it can be expressed in the form $f_x\ y_n + f_{n-1}\ y_{n-1} + - - - - + f_1 y_1 + f_0 = 0$

where fi is an ith order polynomial in x. The general form of this equation is f(x, y) = 0
for example-

$5x + 3y - 15 = 0$

$5x^2 - xy + 4y^2 = 0$

Linear and Non-Linear

Any function of one variable which does not graph as a straight line in two dimensions.

Any function of two variable which does not graph as a plane in three dimensions, can be said to be non-linear.

That is, the function y = f(x) is a linear function f the dependent variable y charges indirect proportion to the change in independent variable y is not direct or exact proportional to the independent variable x, example $y = x^3 + 4$ is a non-linear function.

One the other hand f(x) is said to be non-linear, if the department variable y is not direct or exact proportional to the independent variable x, example: $y = x^3 + 4$ is a non-linear function.

Polynomial Equations

Polynomials: In mathematics, a polynomial is a finite length expression constructed from variables and constant, using the operation and constant non-negative integer exponents.

Example:

a) $x^2 - 4x + 7$ is a polynomial

b) $x^2 - {}^4/_x + 7x^{3/2}$ is not polynomial because its second term involves division by the variable x and also because its third term contain to an exponent that is not an integer.

Polynomial functions polynomial functions is a function defined by evaluating a polynomial. A function f of one, argument is called a polynomial function is it satisfies.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2}$ +--------+$a_2 x_2 + a_1 x^1 + a_0$

where x is non-negative integer and a_0, a1,a_2 -------- an are constant coefficient.

Polynomial Equation:

When an nth degree polynomial is equal to zero, the result is said to be a polynomial equation of degree x ‘that is

$a_n xn + a_{n-1} x^{n-1} + a_{n-2} x^{n-2}$ + ------ + $a_2 x^2 + a_1 x^1 + a_0 = 0$

This is called n^{th} degree polynomial and has n roots. Roots may be real and different, real and repeated, complex numbers. Since complex roots appear in pairs.

If n is odd then the polynomial has at heart one real roots. for example, a cubic equation of the type
$a_3 x^3 + a_2 x^2 + a_1 x + a_0 =$

Will have at heart one real root and the remaining two may be real or complex roots.

Some example are:
$x^3 - 5x^2 + 8x + 6 = 0$
$x^2 + 5x + 6 = 0$

Transcendental Equations:

A non-algebraic equation is called a transcendental equation. These include trigonometric, exponential and loganthmic functions. Example.

i) $x = e^{-x}$

ii) $x = \text{Sin } x$

iii) $\log x - 1 = 0$

iv) $x - e^{1/x} = 0$

v) $x - 2 \sin x = 0$

vi) $5 \tan x - x = 0$

A Transcendental equation may have a finite or an infinite number of real roots or may not have real root at all.

Iterative Methods:

An iterative techonoqves usually begins with an opproxmate vahex of the root, known as the initial guess, which is then successively corrected iteration by iteration. the process of iteration stops when the desired level of cauracy is obtained. Interative methods, based on the number of quesses they use, can be grouped into two categories–

1. Bracketing methods (Interpolation methods)
2. Open end methods (Extrapolation methods)

Bracketing methods:

Start with two initial quesses that ‘pracket’ the root and then systematically reduce the width of the bracket until the solution in reached. Two popular methods under this are:

1. Bisection Method
2. False position method

These methods are based on the assumption that the function changes sign in the vicinity of a root.

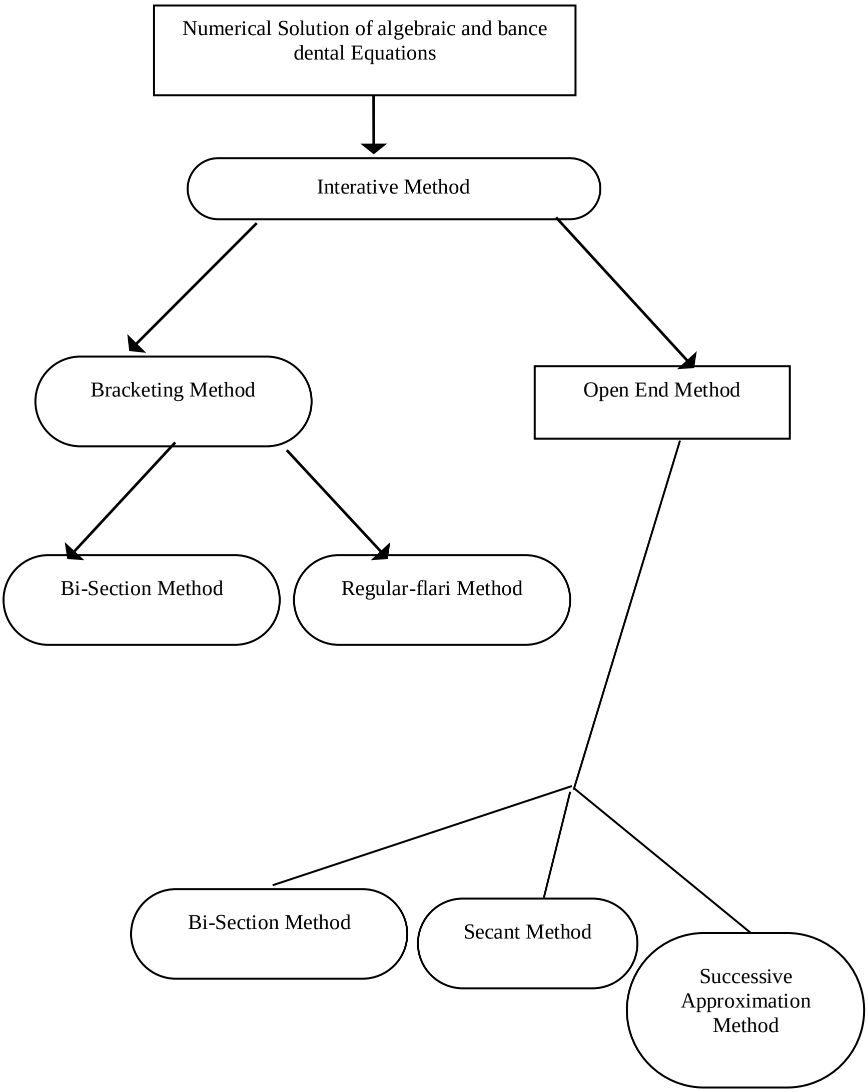
Open End Methods:

Use a single starting value or two values that do not necessarily bracket the root. The following iterative methods fall under this category-

a) Newton Rophson method
b) Secant method
c) Muller’s method
d) Fixed-point method
e) Barstow’s method

It may be noted the bracketing method requires to find sign change in the function during evoxy iteration. open end methods do not requires this.

Method to obtain the roots of Equation



Method of Solving Equations

The polynomial equation and the transcendental equation can be solved by the following method:

i) Graphical method
ii) Bisection method
iii) Newton Rohpson method
iv) false position method
v) Secant method

Only 3 three methods in our syllabus – Bisection, Newton Raphson and false position method.

Bisection (Or Bolzano) Method

One of the first numerical methods originated to find the root of a non-Linear equation f(x) = 0
was the bisection method (also called binary – search method)

where f(x) is a real continuous function has at least one root between a and b y f(a) f(b) < 0

Steps Involved in Bisection Method

1) Choose a and b as two suppose for the root such that f(a) . f(b) < 0 or in other wods f(a) = +ve and f(b) = –ve (opposite part)

2) Approximate the root, x0 of the equation f(x) = 0 as the mid point between a and b as Ist Iteration

$$x_0 = \frac{a + b}{2}$$

This is the first approximation.

3) Now Check the following

i) If f(a) f(x0) < 0 i0. f(0) = +ve and f(x) = –ve (opposite pair) then the roots lies between a and xc then a_1 = a and b_1 x_0

2nd Iteration: $x_1 = \frac{a_1+b_1}{2} = \frac{a+x_0}{2}$

If +(a) + (x0) > a i.e f(a) = +ve and tex = -ve opposite pair) then the x ot lies between b and x_0 then a_1 b and b_1 = x_0

2nd Iteration : $x_1 = \frac{a_1+b_1}{2} = \frac{b+x_0}{2}$

from these any one of them will be second approximation.

iii) if +(a) . f(x_0) = 0 then the root is x_0. stop two algorithm is this is true.

4) Find the new approximation of the root

3rd Iteration: $x_n = \frac{b_x+b_n}{2}$

Which gives us the nth approximation of the root of f(x) = 0 and the roots lies between (a_n, b_n) this procedure an be repeated until the interval containing the root is as small as we hope.

Convergence of Bisection Method:

In Prisection method the internal containing the root is reduced by a factory of 2. The same procedure is repeated for the new internal. If the procedure is repeated n times then the internal containing the root is reduced to the size

$$\frac{x_2 - x_1}{2^n} = \frac{dx}{2^n}$$

After n iterations the root must lie between $\left(+\frac{dx}{2^n}\right)$ of our approximation. This means that the error pound at nth iteration is

$$e_n = \left|\frac{dx}{2^n}\right|$$

Similarly $e_{n+1} = \left|\frac{dx}{2^{n+1}}\right|$

$$e_{n+1} = \frac{1}{2} e_n$$

That is the error decrease linearly with each step by a factory ½. therefore the bisection method is linearly convergent. This is also called rate of convergent.

Since the rate of convergent is show so a large number of iterations may be needed.

Advantage of Bisection Method

The following are the advantage of Bisection Method:

1. The bisection method is always convergent since the method brackets the root the method is guaranteed to converge:
2. As iteration are conducted the internal gets halved. So one can guarantee the error in the solution of the equation.

Example (1)- Perform five iteration of the bisection method to obtain the smallest positive root of the equation

$F(x) = x^3 - 5x + 1 = 0$ (UPTU-2011)

Solution:

We have, $f(x) = x^3 - 5\times 11 = 0$____________(1)

Put x = 0, f(0) = +1 = +ve;

Put .x = 1 f,(1) = 1− 5 + 1 = 3 = −ve

Thence the root lies between 0 and 1 (opposite pair)

+(0) f (1) < 0

I^{st} Approximation $x_1 = \frac{0+1}{2} = \frac{1}{2}$

then $f\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{5}{2} + 1 = -1.375$

put x = $\frac{1}{2}$ in eq^h (1)

Hence the roots lies between f (0) = +ve and f (1/2) = −ve, (opposite pair), then

f (0).f (1/2) < 0

2nd Approximation $x_2 = \frac{0+\frac{1}{2}}{2} = \frac{1}{4}$

Then put x = $\frac{1}{4}$ in equation (1) – we get

$$f\left(\frac{1}{4}\right) = \frac{1}{64} - \frac{5}{4} + 1 = 0.234375$$

= −ve

Hence roots lies between f (0) = +ve and f $\left(\frac{1}{4}\right)$ = −ve. (opposite pair) then

f (0) . f $\left(\frac{1}{4}\right)$< 0

3rd Approximation

$$x_3 = \frac{0 + \frac{1}{4}}{2} = \frac{1}{8}$$

Put x = $\frac{1}{8}$ in equation (1)

$$f\left(\frac{1}{8}\right) = \frac{1}{512} - \frac{5}{8} + 1 = 0.37695$$

= +ve

Hence, the roots lies between $f\left(\frac{1}{4}\right)$ = −ve and $f\left(\frac{1}{8}\right)$ = +ve (opposite pair), then

4^{th} Approximation $x_4 = \frac{\frac{1}{8}+\frac{1}{4}}{2} = \frac{3}{16}$

Put $x = \frac{3}{16} in\ equatioon\ (1), we\ get$

$$+\left(\frac{3}{16}\right) = \left(\frac{3}{16}\right)^3 - \left(\frac{15}{16}\right) + 1 = 0.06909$$

= +ve

Hence roots lies between $f\left(\frac{1}{4}\right)$ = −ve and $f\left(\frac{3}{16}\right)$ = +ve (opposite pair), then

5^{th} Approximation

$$x_5 = \frac{\frac{3}{16}+\frac{1}{4}}{2} = \frac{.7}{32}$$

Root = 7/3.2 **Ans**.

Example – Find the square root of 40 correct up to four decimal places.

Solution: Let

$$x = \sqrt{40}$$

$x^2 = 40$

$f(x) = x^2 - 40 = 0$ ________________________(1)

we know that, $\sqrt{36} = 6$ and $\sqrt{43} = 7$

Hence the real root lies in the internal (6, 7)

By pail, we have f (6.3245) = − 0.00069 (−ve)

So the root lies in the internal (6.3245, 6.3246)

Iteration 1. The fist approximation in

$x_1 = \frac{6.3245+6.3246}{2}$ = 6.32455

Using equation (1), we get

f (6.324.55) = −0.00006 (−ve)

thus the root lies in the internal (6.32455,6.3246)

(opposite pair)

Iteration 2.

The second Approximation is

$$x_2 = \frac{6.32455 + 6.3246}{2} = 6.324575$$

Now f (6.324575) = 0-00000024 (using (1))

= +ve

Thus the root lies in the internal (6.32455, 6.324575)

Iteration 3.

The thir Approximation is

$x_3 = \frac{6.32455+6.324575}{2}$ = 6.3245625

Since x_2 and x_3 both have the same value up to four decimal places.

Thus the required root, correct up to four decimal place in 6.3245625 **Ans**.

Example- Find a positive real root of x – ceas x = 0 by bisection method correct up to four decimal places between 0 and 1. (UPTU-2002)

Solution: at f (x) = x – ceas x = 0 ________(1)

By trail, use have

f (0.7371) = − 0.00332 (−ve)

f (0.7391) = 0.000024 (+ve)

So the root lies in the internal

(0.7371, 0.7391) (opposite pair)

Iteration 1-

f (0.7371). f (0.7391) < 0

$x_1 = \frac{0.7371+0.7391}{2}$ =0.7381

Now put x = 0 0.7381 in equation (1), we get

f (0.7381) = −root <0.7391

Iteration 2-

$x_2 = \frac{0.7381+0.7391}{2}$ = 0.7386

Put x = 0.7386 in equation (1) we get

f (0.7386) = 0.0008 (−ve)

thus the root lies between (0.7386, 0.7381)

f (0.7386). f (0.7391) < 0

Iteration 3-

$x_3 = \frac{0.7386+0.7391}{2}$ = 0.73885

Now put x = 0.73885 in equation (1), we get

f (0.73885) = − 0.00039 (−ve)

thus the root lies in the internal (0.73885, 0.7391)

0.73885 < root < 0.7391

Iteration 4

$x_4 = \frac{0.73885+0.7391}{2}$ = 0.738975

Put x = 0.738975 in equation (1) we get

f (0.738975) = − 0.0001 (−ve)

thus the root lies in the internal (0.738975, 0.7391)

Iteration 5-

$x_5 = \frac{0.738975+0.7391}{2}$ = 0.7390375

f (0.7390375) = − 0.00007 (−ve)

thus the root lies in the internal (0.7390375, 0.7391)

Iteration 6-

$x_6 = \frac{0.739075+0.7391}{2}$ = 0.73906875

Since x5 and x6 both have the same value up to four decimal places.

Thus the required root, correct up to four decimal place is = 0.73906875 **Ans.**

Example- Find a real root of the equation using bisection method $\log_{110}x = 1.2$

Solution-

Let $f(x) \equiv \log_{110}x - 1.2 = 0$ _______________(1)

By trail we have

Put x = 1, f (1) = − 1.2 (−ve)

Put x = 2, f (2) = − 0.59794 (−ve)

Put x = 3, f (3) = 0 − 0.2313637 (+ve)

So that root lies in the internal (2, 3)

(opposite pair)

f (2) – f (3) < 0

Iteration 1-

$$x_1 = \frac{2+3}{2} = 2.5$$

Put x = 2.5 in equation (1) we get

f (2.5) = − 0.2057 (−ve)

thus the root lies in the internal (1.5, 3)

(opposite pair)

Iteration 2-

$$x_2 = \frac{2.5 + 3}{2} = 2.75$$

Now put x = 2.75 in equation (1), we get

f (2.75) = 0.008.1, (+ve)

thus the not lies in the in the internal (2.5, 2.75)

f (2.5) . f (2.75) < 0

Iteration 3-

$$x_3 = \frac{2.5 + 2.75}{2} = 2.625$$

Put x = 2.625 in equation (1), we get

f (2.625) = − 0.09978 (−ve)

thus the root lies in the internal (2.625, 2.75)

f (2.625) . f (2.75) < 1

Iteration 4-

$$x_4 = \frac{2.625 + 2.75}{2} = 2.6875$$

Put x = 2.6875 in equation (1), we get

f (2.6875) = − 0.04612 (−ve)

the roots lies in the internal (2.6875, 2.75)

(opposite pair)

Iteration 5-

$$x_5 = \frac{2.6875 + 2.75}{2} = 2.71875$$

Now put x = 2.71875 in equation (1), we get

f (2.71875) = − 0.0190 (−ve)

thus the root lies in the internal (2.71875, 2.75)

Iteration 6-

$$x_6 = \frac{2.71875 + 2.75}{2} = 2.734375$$

Now f (2.734375) = − 0.0054 (−ve)

Thus the root lies in the internal (2.734375, 2.75)

Iteration 7-

$x_7 = \frac{2.734375+2.75}{2}$ = 2.7421875

Now f (2.7421875) = 0.0013445 (+ve)

Thus the root lies in the internal

(2.734375, 2.7421875)

Iteration 8-

$x_8 = \frac{2.734375+2.7421875}{2}$ = 2.7382812

Now f (2.382812) = − 0.002062 (−ve)

Thus the root lies in the internal

(2.382812, 2.7421875)

Iteration 9-

$x_9 = \frac{2.382812+2.7421875}{2}$ = 2.7402343

Now f (2.7402343) = − 0.0003591 (−ve)

Thus the root lies in the internal

(2.7402343, 2.7421875)

Iteration 10-

$x_{10} = \frac{2.7402343+2.7421875}{2}$ = 2.74121019

Now f (2.7 4121019) = 0.0004926 (+ve)

Thus the root lies in the internal (2.7402343, 2.7412109)

Iteration 11-

$x_{11} = \frac{2.7402343+2.7412109}{2}$ = 2.7407236**Ans**.

Exercise

1) Find a root of the equation x3 – 4x – 9 = 0 correct to three decimal places by using bisection method. (**Ans. 0.7065**)
2) Find a real root of the equation x^4 – 4x – 9 = 0 by using bisection method (**Ans. 2.6875**)
3) Find the real root of the equation x sin x + cos x = 0 between (2, .3) using bisection method. (**Ans. 2.796875**)

4) Solve $x^3 - 9x + 1 = 0$ for the root between x = 2 and x = 4 by the poolzano method (**Ans. 2.9375**)
5) Compute one root of $e^x - 3x = 0$, connect to two decimal places using bisection method (**Ans. 1.51171**)
6) Find a real root of x3 – x= 1 between 1 and 2 by using bisection method (**Ans.1.3247125**) (UPTU-2004)
7) Apply by section method to evaluate $\sqrt{12}$. (**Ans. 3.46411**)
8) Evoluate $\left(1/3\right)^{1/3}$ by using bisection method. (**Ans.0.75**)
9) Find the root of log x = cos x, correct to two decimal places. (**Ans. 1.30**)
10) Find $\sqrt{29}$ by using bisection method. (**Ans. 5.385165**)

2. Regular- Falsi method (Method of false position)

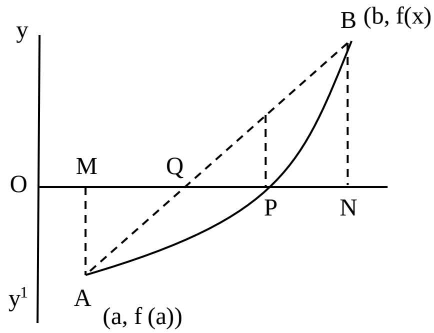
This is the oldest method for finding the real root of an equation and closely resembles the bisection method.

Rule of False Position (Regular Falsi)

Let f (x) = 0 ____________(1)

Let y = f (b) be refornted by the curve AB.

The curve AB cuts the x-axis at P. the real not of (1) is OP.



The false position of the curve AB is taken as the chord AB. The chord AB cuts the x-axis at Q the approximate root of f (x) = 0 is OQ. By this method we find OQ

Let [a, f (a)]. B [b, f (b)] be the extra unites of the chord AB.

The equation of the chord AB is

$y - f(a) = \frac{f(b)-f(a)}{b-a}(X - a)$ (Two pair to form)

To find OQ put y = 0

$$O - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$(x - a) = \frac{-(b-a)f(a)}{f(b)-f(a)}$

$= x - a + \frac{(a-b)f(a)}{f(b)-f(a)}$

$= = \frac{af(b)-af(a)+a\,f(a)-b\,f\,(a)}{f(b)-f\,(a)}$

$$x \quad a\ f\ (b) - b\ f(a$$

If f (a) and f (x_0) are of opposite sign then the root lies between a and x_0 otherwise it lies between x0 and b.

If the root lies between a and x_0 the neat approximation is

$$x_1 = \frac{a\ f(x_0) - x_0 f(a)}{f(x_0) - f(a)}$$

Other wise $x_1 = \frac{x_0\ f(b) - b f(x_0)}{f(b) - f(x_0)}$

The above method is a applied repeatedly till the desired accuracy is obtained.

In general, if we denote a = x0, b = x, then the iteration formula can be written as.

$$X_{n+1} = \frac{X_{n+1} f(x_n) - X_n\ f(X_{n-1})}{f(X_n) - f(X_{n-1})}$$

Rate of Convergence of False Position Method

Example-

The equation $2x^3 + 5x^2 + 5x + 3 = 0$ has a root in the internal [−2, −1]. Starting is the a = −2.0 and b = −1.0

as initial approximations, perform three iterations of the regular falsi method.

Solution: Let f (x) = $2x^3 + 5x^2 + 5x + 3$ ______________(1)

which gives f (−2) = − 16 + 20 – 10 + 3 = −3 (−ve)

f (−1) = − 2 + 5 – 5 + 3 = 1 (+ve)

ie. f (−2) f (−1) < 0

Here a = −2, b = −1

First Iteration

$$X_0 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{-x \times 1 - (-1)(-3)}{1 - (-3)} = \frac{-5}{4} = -1.25$$

Now f (−1.25) = $\frac{-125}{32} + 5 \times \frac{25}{16} + 5 \times \frac{(-5)}{4} + 3$

using (1), = $\frac{21}{32}$ = 0.65625 (+ve)

the root lies in (a_1 x_0) = (-2_1 −1.25)

Second Iteration

$$x_1 = \frac{-2 \times 0.65625 - (-1.25)(-3)}{0.65625 - (-3)} = \frac{-5.0625}{3.65625}$$

= − 1.3846

f (− 1.3846) = 0.353)

the root lies in $(a_1\ x_1) = (-2_1 - 1.346)$

Third Iteration

$$x_2 = \frac{-2 \times 0.3537 - (-1.3846) \times (-3)}{0.3537 - (-3)} = \frac{-4.8612}{3.3537}$$

$= -1.4495$ **Ans.**

Example- Find the real root of x3 – 5x + 3 = 0 by using Regular Falsi Method (UPTU-2004)

Solution- Let

$f(x) = x^3 - 5x + 3$

By trail we get

$f(0.5) = (0.5)^3 - 5 \times 0.5 + 3 = 0.625$ (+ve)

$f(1) = 1^3 - 5 \times 1 + 3 = -1$ (−ve)

so the real lies in the internal (0.5,1)

Let a = 0.5, b = 1

First Iteration

$$x_0 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{af(1) - bf(0.5)}{f(1) - f(0.5)}$$

$$= \frac{(0.5)(-1) - 1(0.625)}{-1 - 0.625} = \frac{1.125}{1.625}$$

= 0.69230

Now, f (0.69230) = − 0.12970

The root lies in (0.5, 0.69230) we get

Second Iteration

$$x_1 = \frac{0.5 \times f(0.69230) - 0.69230 \times f(0.5)}{f(0.69230) - f(0.5)}$$

= 0.65925

f (0.65925) = − 0.009730, using h(1)

the root lies in (0.5, 0.65925) we get

Third Iteration

$$x_2 = \frac{0.5 \times f(0.65925) - 0.65925 \times f(0.5)}{f(0.65925) - f(0.5)}$$

= 0.656808

Using (1)

f (0.656808) = − 0.000694

the root lies in (0.5, 0.656808), we get

Fourth Iteration

$$x_3 = \frac{0.5 \times f(0.656808) - 0.656808 \times f(0.5)}{f(0.656808) - f(0.5)}$$

= 0.656634

Here $x_2 = x_3$ (Up to three decimal places)

Hence the required real root in 0.6566**Ans.**

Example- Find the real root of the equation $xe^x = \cos x$ in the internal (0,1) by using Regular Falsi method correct to four decimal places. (UPTU-2004, 2005)

Solution: We have $ne^x = \cos x$

$\Rightarrow \cos x - xe^x = 0$

Let $f(x) = \cos x - xe^x$ _______________ (1)

Here for less iteration, we choose-

a = 0.5 and b = 0.6 both are between (0,1)

from equation (1) $f(0.5) = \cos(0.5) - 0.5\, e^{0.5} = -0.05322$ (−ve) and

$f(0.6) = \cos(0.6) - 0.6\, e^{0.6} = -0.26793$ (+ve)

so the real root lies in the internal (0.5, 0.6)

First Iteration

$$x_0 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.5 \times f(0.6) - 0.6\, f(0.5)}{f(0.6) - f(0.5)}$$

$$= \frac{(0.5)(0.26793) - 0.6\,(-0.005322)}{0.26793 - (-0.05322)}$$

= 0.516572

Now from (1) f (0.516572) = − 0.0036

The root lies in (0.516572, 0.6) we get

Second Iteration

$$x_1 = \frac{0.516572 \times f\ (0.6) - 0.6 \times f(0.516572)}{f(0.6) - f(0.516572)}$$

$$= \frac{0.516572 \times (0.26793) - 0.6(-0.0036)}{0.26793 + 0.0036}$$

= 0.517678

Now f (0.517678) = − 0.000241

⇒ the root lies in (0.51768, 0.6) we get

Third Iteration

$$x_2 = \frac{0.516578\ \times f\ (0.6) - 0.6\ \times f(0.516578)}{f(0.6) - f(0.516578)}$$

$$= \frac{0.516572 \times (0.26793) - 0.6(-0.000241)}{0.26793 + 0.000241}$$

= 0.5166469

Now f (0.5766469) = 0.003375 (using (1))

The root lies in (0.5166469, 0.517678) we get

Fourth Iteration

$$x_3 = \frac{0.5166469 \times (-0.000241) - (0.517678)(0.00355)}{-0.000241 - 0.003375}$$

= 0.5176032

Now f (0.5176092) = 0.00045 (using (1))

The root lies in (0.5176092, 0.517678) we get

Fifth Iteration

$$x_4 = \frac{0.5176092 \times (-0.000241) - (0.517678)(0.00355)}{-0.000241 - 0.0045}$$

= 0.517654

Here, x3 = x4 (up to four decimal places)

Hence the required real root is = 0.517654 = 0.5177 **Ans**.

Example- Find the root of the equation 2x – $\log_{10} \times$ x = 7 which lies between 3.5 and 4 correct to five places of decimal. Using method False position.

Solution- 2x – $\log_{10}$ x = 7

⇒ 2x – $\log_{10}$ x = 7

Then let, f (x) = 2x – $\log_{10}$ x – 7 ______________(1)

f (4) = 8 – $\log_{10}$ 4 – 7 = 1 – 0.60206 = 0.39794

f (3.5) = 7 – $\log_{10}$ 3.5 – 7 = – 0.54407

the root x_1 lies between 3.5 and 4

By false position method

First Iteration

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{3.5f(4) - 4f(3.5)}{f(4) - f(3.5)}$$

$$= \frac{3.5(0.39794) - 4(-0.54407)}{0.39794 — (0.5440)} = \frac{3.56907}{0.94201}$$

= 3.78878

Hence f (3.78878) = 7.57756 – 0.57850 – 7 (using (1))

The roots x2 lies between 3.78878 and

Second Iteration

Again applying false position method

$$x_2 = \frac{3.78878\, f(4) - 4f(3.78878)}{f(4) - f(3.78878)} = \frac{1.51147}{0.39888}$$

= 3.78928

Hence the root given equation is 3−78928.

Exercise

Solve the following equation by Regular Falsi Method

1) x3 + x^2 – 3x – 3 = 0 (Root between 1 and 2) (**Ans. 1.728**)
2) x^3 – 4x + 1 = 0 (**Ans. 0.25**)'
3) x3 −5 x + 7 = 0 (Root between 2 and 3) (**Ans. 2.7472**)
4) xe^x – 3 = 0 (**Ans.1.0498**)
5) $3x^2$ + 5x = 40 (**Ans.2.138**)
6) x2 – 1 = 0 (**Ans. 0.9990**)
7) Find the real root of the equation cos x = $\sqrt{x}$, by method of false position correct up to five decimal place. (**Ans.0.64171**)
8) Complete the root of the equation x = log 2 (x + 1) by Regular Falsi Method correct to three decimal places (**Ans. 1.490**)
9) Find the real root of the equation x log10 x – 1.2 = 0 correct to five decimal places by Regular falsi method (**Ans.2.74065**) (UPTU-2004)
10) Use the method of false position to find the root of the equation x3 – 18 = 0 given that it lies between 2 and 3 write down three steps of the procedure. (**Ans. 2.621**)

Newton-Raphson Method (Newton Method)

This method is used to obtain a better approximation of a root using the earlier approximations obtained by Bisection method or Regular falsi method.

Let x_0 be an approximate roots of f (x) = 0

Let $x_1 = x_0 + h$ be the correct root so that ________(1)

$f(x_0 + h) = 0$

To find h

For this $f(x_0 + h)$ by toylor's sones

$f(x_0 + h) = f(x_0) + h f^1(x_0) - 1\frac{h_2}{L_2} f^{11}\left(x\frac{1}{0}\right) + -----$

$0 = f(x_0) + h f_1(x_0)$ (∵ $f(x_0 + h) = 0$

$h = \frac{f(x_0)}{f^1(x_0)}$ ____________(2) [Neglecting the second and higher order derirative]

But $x_1 = x_0 + h$ ________(3)

Putting the value of h from (2) in equation (3)

$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$

x in better approximation than x_0

then

$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$

x_2 is better approximation than x_1

successive approximation ore . x_3, x_4 --- x_{n+1}

then

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

which Newton – Rapohon method

Note:

1) This method is applicable only when his very small i.e f′(x) is large ‘mother words when the graph of f (x) near the root is nearly vertical.
2) Newton-Rapshon method is the best known procedure for finding the roots of an equation.
3) It is applicable to the solution of all types of equation i.e., algebraic and transcentral and also useful for calculating complete roots.
4) This formula converge rotidly . If the initial approximation x_0 is taken using to the root x.

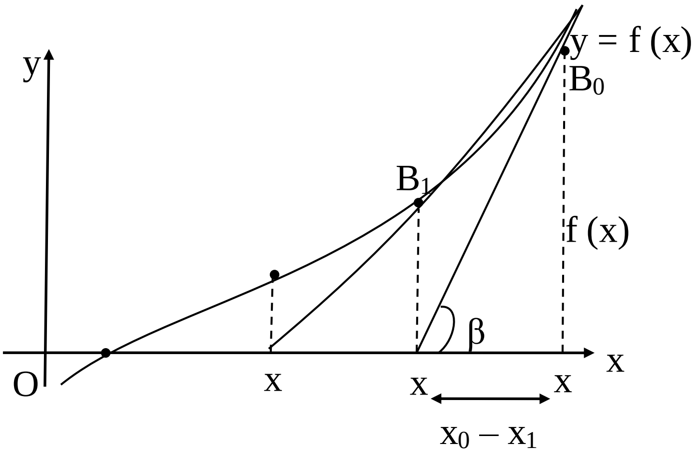
Geometrical Interpretation

This method is equivalent to replacing a small arc of the curve y = f(x) by a tongent line drawn at a point of the curve. Drawn +a tangent to the curve at B_0 which meet x-axis at x_1 then draw a tangent at B_1 which meet x-axis at x_2 and so on.

y = f (x)

$\therefore$ slope of tangent at $B_0 = \frac{dy}{dx} = \acute{f}(x_0)$

$$\tan\beta = \acute{f}(x_0) = \frac{f(x_0)}{x_0 - x_1}$$



$$x_0 - x_1 = \frac{f(x_0)}{\acute{f}(x_0)}$$

$$x_1 - x_0 = -\frac{f(x_0)}{\acute{f}(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{\acute{f}(x_0)}}$$

In general

$$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)}$$

Rate of Convergence of Newton-Roplnson Method

Let x_r be the exect value of the root and x_n be the value of root after n iteration (approximation) then error en is

$e_n = x_n - x_r \Rightarrow x_n = x_r + e_n$ ____________(1)

$\therefore x_{n+1} = x_r + e_{n+1}$ _______________(2)

We know that

$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)}$

Now using (1) $x_r + e_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)}$

Using the value of x_n in above we get

$x_r + e_{n+1} = (x_r + e_n) - \frac{f(x_r+e_n)}{\acute{f}(x_r+e_n)}$ using (1)

$\Rightarrow e_{n+1} = e_n - \frac{f(x_r+e_n)}{\acute{f}(x_r+e_n)}$ ____________(2)

By Taylor's theorem we have

$f(x + h) = f(x) + h f^1(x) + \frac{h^2}{L^2} f^{11}(x)$ 7 ------------

$f(x_r + e_n) = f(x_r) + e_n f^1(x_r) + \frac{e_n^2}{L^2} f^{11}(x_r)$ -----------

$e_n f^1(x_r) + \frac{e_n^2}{L^2} f^{11}(x_r) +$ ---------------------

{As f (x$_r$) = 0 (esact volw)}

and $f^1(x_r + e_n) = f^1(x_r) + e_n f^{11}(x_r) +$ ---------------

Putting these value in equation (3) we get

$$e_{n+1} = e_n - \frac{[e_n f^1(x_r) + e_n^2 f^{11}(x_r) + ---]}{[f^1(x_r) + e_n f^{11}(x_r) + ---]}$$

$$= \frac{e_n f^1(x_r) + e_n^2 f^{11}(x_r) - e_n f^1(x_r) - \frac{e_n^2}{L^2} f^{11}(x_r)}{f^1(x_r) + e_n f^{11}(x_r) \pm ---}$$

$$= \frac{e_n^2 f^{11}(x_r)}{2[f^1(x_r) + e_n f^{11}(x_r)]}$$

$$= \frac{e_n^2 f^{11}(x_r)}{2f^1(x_r)} \frac{1}{\left[1 + \frac{e_n . f^{11}(x_r)}{2f^1(x_r).}\right]}$$

Since $en. \frac{f^{11}(x_r)}{2f^1(x_r)}$ is very small so

neglect this terms, we get

$$\boxed{e_{n+1} = e_n^2 \frac{f^{11}(x_r)}{2f^1(x_r)}}$$

which shows that at each iteration about error is proportional to the square of the previous error and hence we may say that the convergence is quadratic.

Hence if at the First iteration, we have an answer correct to one decimal places them it should be correct to the two places at the second iteration. To four place at the third iteration etc. It means that the no. of correct decimal places at each iteration is doubled almost. Hence this method converges very rapidally.

Working Rule to Solve f (x) = 0 By Newton

Raphson Method

Step 1-Taking two close number b and c such that f (b) and f (c) are of opposite sign. Then the root $\propto$ lies between b and c

Step 2- Out of f (b) and f (c) choose which is nearer to zero. If f (b) is nearer to zero then b is an initial approximation root (x_0) of the given equation.

Step 3- Apply Newton Raphson formula to find out better approximate root x_1.

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

Repeat the process to get successive approximates

Step 4- Stop the process when two approximate root are equal.

Example-Find the smallest positive root of the equation $x^3 - 2x + 0.5 = 0$

Solution

$f(x) = x^3 - 2x + 0.5$

$f(0) = 0.5$

$f(0.1) = 0.001 - 0.2 + 0.5 = 0.301$ (+ve)

$f(0.3) = 0.027 - 0.6 + 0.5 = -0.073$ (−ve)

Hence the root lies between 0.1 and 0.3 (opposite pair)

$f(0.3) = 0.073$ is nearer to zero

so 0.3 is first approximation we have a better approximation as (0.3 + h)

By Newton – Rapshon Method

$f^1(x) = 3x^2 - 2$

$f^1(0.3) = 3(0.3)^2 - 2 = -1.73$

Second Approximation Root

$$x_2 = 0.3 - \frac{f(0.3)}{\acute{f}/_{0.3}}$$

$$= 0.3 - \frac{-0.073}{-1.73} = 0.3 - \frac{73}{1730}$$

$= 0.3 - 0.0422$

$= 0.2578$ **Ans.**

Example- Find a positive root of x4 – x = 10 using Newton Raphson Method

Solution

Let $f(x) = x^4 - x - 10$ ____________(1)

$f^1(x) = 4x^3 - 1$ ____________(2)

By trail in equation (1)

$f(1) = -10$ (−ve)

$f(2) = 4$ (+ve)

Thus the root lies in the interval (1, 2)

(opposite pair)

so x_0 = first approximation = 2 in near to zero.

Let $x_0 = 2$ be an approximation value of the root

we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} \text{ ____________(1)}$$

By putting n = 0, 1, 2 – in equation (3) we get successive iteration as follows.

First Iteration for n = 0

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 2 - \frac{2^4 - 2 - 10}{4 \times (2)^3 - 1} \text{ using (1) , (2)}$$

$$= 2 - {}^4\!/_{32} = 1.871$$

Second Iteration for n = 1

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 1.871 - \frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1}$$

= 1.8558 ≈ 1.856

Third Iteration for n = 2

$$x_3 = x_2 - \frac{f(v_3)}{f^1(x_3)} = 1.856 - \frac{(1.856)^4 - 1.856 - 10}{4(1.856)^3 - 1}$$

= 1.85558 ≈ 1.856

Here, $x_2 - x_3$ (correct up to three places of decimal)

Hence the required root is x = 1.856 **Ans**.

Example – Apply Newton Raphson Method to solve

3x – cos x – 1 = 0

Solution-f (x) = 3x – cos x – 1 = 0

$f^1(x) = 3 + \sin x$

f (0) = 0 − 1−1= −2

f (0.6) = 1.8− cos 0.6 – 1 = − 0.0253

f (1) = 3 cos 1 – 1 = 1.4597

As f (0.6) in nearer to zero then f (1). So ase take first approximate root as 0.6 By Newton –Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$$x_1 = x - \frac{3x - \cos x(-)}{3 + \sin x}$$

$= 0.6 - \frac{3(0.6) - \cos 0.6 - 1}{3 + sin 0.6}$

= 0.6071

and x2 = 0.6071 $- \frac{3(0.6071) - \cos 0.6071 - 1}{3 + sin 0.6.71}$

= 0.6071

since $x_2 = x_1$, therefore the fealroot of the given equation is 0.6071.

Example- write Newton Raphson Procedure for finding $\sqrt[3]{N}$. Who N is a number Use it to find $\sqrt[3]{N}$ correct to 2 decimal assuming 2.5 as the initial approximation.

Solution

Let x = $\sqrt[3]{N}$

⇒ $x^3 = N$ ⇒ $x^3 - N = 0$

Let $f(x) = x^3 - N = 0$

⇒ $f^1(x) = 3x^2$

By Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$ n = 0,1, 2 ________(1)

Putting N = 18,

x_0 = Approx cube root of 18

= 2.5 in equation (1) we get

$x_1 = \frac{2(2.5)^3+18}{3(2.5)^2}$ = 2.62667**Ans**.

Example-By iteration method find the value of $(48)^{1/3}$ connect to three decimal places.

Solution

Let x = $\sqrt[3]{N} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

Let $f(x) = x^3 - N = 0$

$f^1(x) = 3x^2$

By Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2}$ n = 0, 1, 2---------

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + n}{3x_n^2}$$

$x_{n+1} = \frac{2x_n^3+N}{3x_n^2}$ n = 0, 1, 2 --- (1)

Putting N = 48

x0 = aproxx cubic root of 48 = 3.5 in equation 1 we get

$$x_1 = \frac{2(3.5)^3 + 48}{3\ (3.5)^2} = \frac{133.75}{36.75}$$

= 3.6395

$x_2 = \frac{2(3.6395)^3+48}{3(3.6395)^2} = \frac{148.+173}{39.7379}$ = 3.6342

$x_3 = \frac{2(3.6342)^3+48}{3(3.6342)^2} = \frac{143.9967}{39.62222}$ = 3.635

as $x_2 = x_3$, 60

$$(48)^{1/2} = 3.6342$$

Correct up to 4 decimal places

Hence $(48)^{1/2} = 3.6342$**Ans**.

Example- Find an iteration formula to find $\sqrt{N}$ (where N is a positive numbers) and hence find

(a) $\sqrt{5}$, correct to 5 decimal places

(b) Find $\sqrt{24}$

Solution-

Let x be the square root of a given positive numbers N.

$x = (N)^{1/2} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0$

Let $f(x) = x^2 - N = 0$

$f^1(x) = 2x$

By Newton – Rophson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$x_{n+1} = \frac{x_n^2+N}{2x_n}$ __________(1)

n = 0, 1, -----

It is the Iterative formula to find $\sqrt{N}$

a) $\sqrt{N}$⇒Let N = 5

x_0 = Approx root of 5 = 2

$x_1 = \frac{(2)^2+5}{2\times2}$ = 2.5 using (1)

$x_2 = \frac{(2.25)^2+5}{2\times(2.25)}$ = 2.23611 **Ans**.

b) $\sqrt{24}$, Let N = 24

x0 = Approx root of 24 = 5

using (1)

$x_1 = \frac{(5)^2+24}{2\times5}$ = 4.9

$x_2 = \frac{(4.9)^2+24}{2\times4.9}$ = 4.899

$x_3 = \frac{(4.889)^2+24}{2\times(4.889)}$ = 4.89898 **Ans**.

Exercise

Using Newton Raphson Method find a real root of the following equation.

1) $x^3 - 6x + 4 = 0$ **(Ans. 0.7321)**
2) $x^3 - 5x + 3 = 0$ **(Ans. 0.6566)**
3) $x - e^{-x} = 0$ **(Ans. 0.5671)**
4) $x^2 + 4 \sin x = 0$ **(Ans. −1.9338)**
5) $x^5 + 5x + 1 = 0$ **(Ans. − 0.1999)**
6) $2x - \log_{10} x - 7 = 0$ **(Ans. →3.7893)**
7) $\tan x = 4x$ **(Ans. 0.0000)**
8) Find $\sqrt{12}$, correct to five decimal places by Newton- Raphson Method. **(Ans.→3.46410)**
9) A sphere of wood 2 cm in diameter floating in water sinks to a depth d given by the equation $d^3 - 3d^2 + 2.5 = 0$. Find d correct to two decimal places **(Ans. 1.17)**
10) Find the real root four decimal of the equation $x^6 - x^4 - x^3 - 1 = 0$ lying between (1, 2) **(Ans. 1.4036)**
11) A root of the equation $x^3 - x^2 - x + 1 = 0$ is to be determined by the Newton Raphson Method. The initial approximation to the root is given as 0.9. Find the root of the equation **(Ans.→ 1.0001)**
12) The value of $N^{1/5}$, N > 0 is to determined numerically, constant the Newton Raphson scheme to find the required root. Apply it to find $(29)^{1/5}$, correct to three decimal places, starting with the initial approximation as 2.0.

Interpolation

One to the basic ides in mathematics in that of a function and most useful tool of numerical analysis is interpolation. According to the it- “Interpolation is the art of reading between the lines of the table.” Broadly speaking interpolation is the problem of obtaining the value of function for any given function information about it.

In general if g is a function of x then the functional relation may be denoted by the equation y = f(x). there x is the independent variable and y is the dependent variable. It is usual to call x as argument and y as function of the argument (or entry).

Let y I f (x) for a set of value of x

x	x_0	x_1	x_2	-- x_n
y	y_0	y_1	y_2	-- y_n

Thus the process of finding the values of y corresponding to any value of x = xi between x_0 and x_n is called interpolation.

“Interpolation is the technique of estimating the values of a function for any intermediate value of the argument when the value of the function corresponding to a number of the value of argument are give”.

“When the values are to be estimated for the past period or future period i.e., outoide the given range is called extrapolation,

Methods of Interpolation

Broadly speaking there are two types of methods of interpolation-

i) Graphic Method

ii) Algebric Method

However algebraic method can be further classified into following methods-

a) Methods for equal intervals
b) Methods for unequal interval
c) Methods for central differences
d) Methods of heart square of curve fitting

Finite Differences

Let y = f (x) be a describe function.

If x_0, (x_0 + h), (x_0 + 2h), (x_0 + 3h) --- (x_0 + nh) are the successive value of x, where four conseutive value after by a quantity h then the corresponding value of y are y_0, y_1, y --- yn.

To determine the value of f (x) or f (x) for some intermediate arguments the following three type of difference are found useful-

i) Forward difference
ii) Backward difference
iii) Central difference

1. **Forward Differences**

Let y = f (x) be a function at equal spaces of x, say at x = a, (a + h) ------ (a + 2h)--- etc.

Then the corresponding value of f (x) are.

f (a), f (a+h), f (a + 2h) --- etc. here x is argument and y is entry.

The difference f (a + h) – f (a) is known as

First forward difference of f (x) at x = a and it is denoted by Δ f (a)

Therefore, the first difference of f (x) at x = a is

$\Delta f(a) = f(a + h) - f(a)$

In general it is defined as

$$\boxed{\Delta f(x) = f(x + h) - f(x)}$$

Where Δ is called forward difference opoxrter similarly.

$\Delta^2 f(x) = \Delta f(x + h) - \Delta f(x)$

$\Delta^3 f(x) = \Delta^2 f(x + h) - \Delta^2 f(x)$

$$\boxed{\Delta^n f(x) = \Delta^{n-1} f(x + h) - \Delta^{n-1} f(x)}$$

Note:

1) Δ f (x) means f (x) is to be subtracted from root entry
2) The difference f (x + h) – f (x) is denoted by placing Δ before the second entry.
3) Δ^2 is not the square of the operator Δ but Δ^2 means Δ of xrated by Δ.
4) Δ [f (x) ± g (n)] = Δ f (x) ± Δ g (x). i.e. Δ is lines
5) Δ C = 0 (Difference of constant function are zero)

Difference Table (Forwarded difference Table)

Amount x	Entry f (x)	First difference Δ(fx)	Second difference Δ^2 f(x)	Third difference Δ^3 f(x)
a	f (a)			
		f (a + h) f (a) = Δ f (a)		
a + h	f (a + h)		Δ f (a + h) − Δ f (a) = Δ^2 f (a)	
		f (a + 2h)− f (a + h) = Δ f (a + h)		Δ^2 f (a + h) − Δ^2 f (a) = Δ^3 f (a)
a + 2h	f (a + 2h)		Δ f (a + 2h) − Δ f (a + h) Δ f (a + h)	
		f (a + 3h) – f (a + 2h) = Δ f (a + 2h)		
a + 3h	f (a + 3h)			

The first entry t (cd) is called the leading terms

Example – Given f (0) = 3, f (1) = 12, f (2) = 81, f (3) = 200, f (4) = 100, and f (5) = 8 from a difference table and find Δ5 f (10).

Solution- The difference table is as follow-

x	f (x)	Δ f (x)	Δ^2 f (x)	Δ^3 f (x)	Δ^4 f (x)	Δ^5 f (x)
0	3}	9	60	−10	−259	755
1	12}	69	50	−269	496	
2	81}	119	−219	227		
3	200}	−100	8			
4	100}	−52				
5	8}					

x	f (x)	Δ f (x)	Δ^2 f (x)	Δ^3 f (x)	Δ^4 f (x)	Δ^5 f (x)
0	3}	9	60	−10	−259	755
1	12}	69	50	−269	496	
2	81 }	119	−219	227		
3	200 }	−100	8			
4	100 }	−52				
5	8}					

Hence from table, Δ^5 f (0) = 755 **Ans**.

Example- If

x	1	2	3	4	5
y	2	5	10	20	30

Find by forward difference table Δ^4 y (1)

Solution-

x	y = f (x)	Δ f (x)	Δ^2 f (x)	Δ^3 f (x)	Δ^4 f (x)
1	2}	3	2	2	− 8
2	5}	5	5	−5	
3	10	10	0		
4	20	10			
5	30				

From table, we obsance that

$\Delta 4\ f(1) = -8$ **Ans**.

2. **Backward (descending) difference**

Let y = f (x) be given at equal spaces of the independent variable x at x = a, a + h, a + 2h --- etc. then the backward difference is denoted by $\nabla f(a)$ and defined as

$\nabla f(a) = f(a) - f(a-h)$

In general

$$\boxed{\nabla f(x) = f(x) - f}$$

Where

∇ (del) is called backward difference

Second backward difference is denoted by $\nabla^2 f(x)$ and

$\nabla 2\ f(x) = \nabla [\nabla f(x)]$

$= \nabla [f(x) - f(x-h)]$

$= \nabla f(x) - \nabla f(x-h)$

$= [f(x) - f(x-h)] - [f(x-h) - (f\ x - 2h)]$

$= f(x) - 2 f(x-h) + f(x-2h)$

Similarly

$\nabla^3 f(x) = f(x) - 3 f(x-h) + 3 (x-2h) - f(x-3h)$

In General

$\nabla^n f(n) = \nabla^{n-1} [\nabla f(x)]$

$= \nabla^{n-1} [f(x) - f(x-h)]$

$$\boxed{\nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1}}$$

Notes:

1. $\nabla [f(x) \pm g(x)] = \nabla f(x) \pm \nabla f(x)$, i.e ∇ is linear operator.
2. $\nabla [\propto f(x)] = \propto . \nabla f(x)$, $\propto$ being a constant
3. The backward difference f (a) – f (a − h) is denoted by placing backward difference operator ∇ before the first entry.

Example- Constant a backward difference table f (1) = 4, f (2) = 8, f (3) = 12, f (4) = 18, f (5) = 36 find ∇^4 f (5).

Solution- The difference table is as follows-

x	f (x)	∇ f (x)	∇^2 f (x)	∇^3 f (x)	∇^4 f (x)
1	4				
2	8	4			
3	12	4	0		
4	18	6	2	2	
5	36	18	12	10	8

from the table,

we get

∇^4 f (5) = 8. **Ans**.

Table of Backward Difference

Amount x	Entry f (x)	First difference ∇ (fx)	Second difference ∇^2 f(x)	Third difference ∇^3 f(x)
a	f (a)			
		f (a + h) f (a) = ∇ f (a + h)		
a + h	f (a + h)		∇ f (a +2 h) − ∇ f (a + h) = ∇^2 f (a + 2h)	
		f (a + 2h) − f (a + h) = ∇ f (a + 2h)		∇^2 f (a +3 h) − ∇^2 f (a + 2h) = ∇^3 f (a + 3h)
a + 2h	f (a + 2h)		∇ f (a + 3h) − ∇ f (a + 2h) Δ f (a +3 h)	
		f (a + 3h) − f (a + 2h) = ∇ f (a + 3h)		
a + 3h	f (a + 3h)			

Some Others Difference Operators

1. **Shift Operator (E)**

If h is the internal of differencing in the argument x then the operator E is defined as-

E f (x) = f (x + h)

It is also called translation operator due to the reason that is result the next value of the

Function-

$E^2 f(x) = E[E f(x)]$

$= E f(x + h)$

$= f(x + 2h)$

Inverse operator E^{-1} is

$E^{-1} f(x) = f(x - h)$

$E^{-2} f(x) = f(x - 2h)$

$E^{-n} f(x) = f(x - n h)$

In general

$E^n f(x) = f(x + nh)$ for any real n.

Note: If y_0, y_1 ----- y_n are the consecutive value of y_x then

$Ey_0 = y_1$, $E^2 y_3 = y_5$ --- $E^v y_n = y_{n + k}$

2. **Average Operator (μ)**

The average operator μ n defined as-

$\mu f(x) = \frac{1}{2} [f(x + h/2) + f(x - h5)]$

i.e,

$\mu y_x = \frac{1}{2} [y_{(x + h/2)} + y_{(x - h/2)}]$

3. **Differential Operator (D)**

The differential operator D is defined as

D f(x) = $\frac{d}{dx} f(x)$

In general

D^n f(x) = $\frac{d^n}{dx^n} f(x)$

4. **Central Difference Operator (δ)**

$$\delta_{f(x)} = f\left(x + \frac{h}{2}\right)$$

δ is called as the central difference operator

Relation Between Operators

i**. Relation between Δ and E**

Δ f(x) = f(x + h) – f(x)

= E f(x) – f(x)

= (E − 1) f(x)

$\Delta = E -$ or $E = 1 +$

ii**. Relation Between E and ∇**

∇ f(x) = f(x) – f(x − h)

= f(x) − E^{-1} f(x)

= (1 − E^{-1}) f(x)

$\nabla = 1 -$ or $E^{-1} = 1 - \nabla$

or $E = (1 - \nabla$

iii. **Relation between E and δ**

δ f(x) = f(x + h/2) – f(x –h/2)

$= E^{1/2} f(x) - E^{-1} f(x)$

$\delta f(ex) = \left(E^{\frac{1}{2}} - E^{-1/2}\right) f(x)$

∴ $\boxed{\delta}$

$$\delta = E^{1/2}(1 - E^{-1}) = E^{1/2}\nabla$$

and $\delta = E^{-1/2}(E - 1) = E^{-1/2}\Delta$

∴ $\boxed{\delta = E^{1/2}\,\nabla}$

iv. **Relation between E and μ**

$$\mu f(x) = \frac{1}{2}\left[f\left(x + {}^{R}/_{2}\right) - f\left(x - {}^{R}/_{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$= \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$ f(x)

$$\boxed{\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)}$$

v. **Relation between Δ and δ**

$$\Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1 + \frac{\delta^2}{4}}$$

Where δ and μ are entail difference and average operator.

Solution:

$\delta f(x) = f(x + h/2) - f(x - h/2)$

$\delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$

$\delta = E^{1/2} - E^{-1/2}$ ____________________(1)

Squaring we get

$\delta^2 = E + E^{-1} - 2$

$E\,\delta^2 = E^2 + 1 - 2E$

$E\,\delta^2 = (E - 1)^2$

$(1 + \Delta)\,\delta^2 = \Delta^2$

$\Delta^2 - \delta^2\,\Delta - \delta^2 = 0$

then

$$\Delta = \frac{\delta^2 \pm \sqrt{\delta^4 + 4\delta^2}}{2}$$

$$\boxed{\Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1 + \frac{\delta^2}{4}}}$$

vi. **Relation between Δ, ∇, μ and δ**

a. $\mu\delta = \frac{1}{2}(\nabla + \Delta)$
b. $\mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$

Solution- we have

$$\mu f(x) = \frac{1}{2}\left[f\left(x + {h}/{2}\right) + f\left(x - {h}/{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$ ______________(1)

But $\delta = E^{1/2} - E^{-1/2}$ ______________(2)

Multiplying equation (1) and (2), we get

$$\mu\delta = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]\left[E^{1/2} - E^{-1/2}\right]$$

$$= \frac{1}{2}[E - E^{-1}]$$

$$= \frac{1}{2}[1 + \Delta + \nabla - 1]$$

$$\boxed{\mu\delta = \frac{1}{2}(\Delta + \nabla)}$$

from above $\delta\mu = \frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(\nabla + \Delta)$

$$= \frac{1}{2}(\Delta E^{-1} + \Delta)$$

$$\boxed{\delta\mu = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}}$$

vii. **Relation between μ and δ**

$$\mu^2 = 1 + \frac{1}{4}\delta^2$$

We know that

Average operator $\mu f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

$$\mu f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

Squaring both side, we get

$$\mu^2 f(x) = \frac{1}{4}\left[E^{1/2} + E^{-1/2}\right]^2 f(x)$$

$$\mu^2 f(x) = \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4\right] f(x)$$

$$= \frac{1}{4}[\delta^2 + 4]f(x)$$

$$\mu^2 f(x) = \left[1 + {\delta^2}/{4}\right] f(x)$$

$$\boxed{\mu^2 = 1 + {\delta^2}/{4}}$$

Finite Difference Operator

Operator	Definition	Name of the Operator
E f (x)	f (x + h)	The shift operator
Δ f (x)	f (x + h) – f (x)	The forward difference operator
∇ f (x)	f (x) – f (x − h)	The backward difference operator
δ f (x)	f (x + h/2) – f (x – h/2)	This central difference operator
μ f (x)	$\frac{1}{2}\left[f\left(x + {h}/{2}\right) + f\left(x - {h}/{2}\right)\right]$	The average operator

Example-

Evaluate

1. $\Delta \tan^{-1} x = \tan^{-1}(x + h) - \tan^{-1} x$

   $$= tan^{-1}\left[\frac{x + h - x}{1 + (x + h)x}\right]$$

   $= tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$**Ans**.
2. Δ2 cos 2x = Δ [cos 2 (x + h) – cos 2x]
   = Δ cos 2 (x + h) − Δ cos 2 x
   = [cos 2 (x + 2h) – cos2 (x + h)] – [cos 2 (x + h) – cos 2 x]
   = − 2 sin (2x + 3h) sin h + 2 sin (2 x + h)
   = − 2 sin h [sin (2x + 3h) – sin (2 x + h)]
   = − 2 sin h [2cos (2x + 2h) sin h]
   = − 4 sin2 h . cos (2x + 2h)**Ans**.
3. $\Delta^2 (e^{ax + b}) = \Delta \; e^{(x+h)}$

   $= \Delta (e^{a(x+h) + b} - e^{ax+b})$

$= \Delta\ e^{ax+b}\ (e^{ah-1})$

$= (\ e^{a(x+h)+b} - e^{ax+b})\ (e^{ah-1})$

$= e^{ax+b}\ (e^{ah} - 1)\ (e^{ah} - 1)$

$= e^{ax+b}\ (e^{ah} - 1)^2$ **Ans**.

4)

$$\frac{\Delta^2}{E} x^2 = \frac{(E-1)^2}{E} x^3$$

$$= \left[\frac{E^2 + I - 2\ EI}{E}\right] x^3$$

= [E + I2 E–1 – 2J]x3

= Ex3 + I2 E–1x3 – 2Jx3

$= (x + 1)^3 + (x - 1)^3 - 2x^3$

$(x^3 + 1 + 3x^2 + 3x) + (x^3 - 1 - 3x^2 + 3x) - 2x^3$

= 6x

5) Evaluate Δ3 (1 – x) (1 – 2x) (1 – 3x), the internal of differencing being unity.

Solution: We have f(x) = (1 – x) (1 – 2x) (1 – 3x)

$= -6x^3 + 11x^2 - 6x + 1$

i.e. f(x) is a polynomial in x of degree 3.

$\Delta^3\ f(x) = \Delta^3\ (-6x^3 + 11x^2 - 6x + 1)$

$$= -6\ \Delta^3 x^3 + 11\Delta^3 x^2 - 6\Delta x + \Delta^3$$

$$= -6\ \Delta^3 x^2 + 0 - 0 + 0 \begin{pmatrix} \Delta^3 x^2 = 0 \\ \Delta^3 x = 0 \end{pmatrix}$$

= –6(31)

= –36

Exercise

1) Construct a table of forward difference for the following data Evaluate Δ^4 y(0)

X	0	1	2	3	4
y	1.0	1.5	2.2	3.1	4.6

2) Construct a forward difference table and find Δ^4 f(1) if f(1) = 1, f(2) = 3, f(3) = 8, f(4) = 15, f(5) = 25 Ans: 2

3) Construct a table of backward difference for the following data and find Δ^3 f(5)

x	0	1	2	3	4	5
y	2	9	28	65	126	217

Ans: 6

4) from the table of backward difference of the function given below and find Δ^4 f(5).

$f(x) = x^3 - 3x^2 - 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$

Ans: 0

5) from a table of backward difference for the function

$f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$

find Δ^4 f(5)

Ans: 0

6) Evaluate $\Delta^2\left(\frac{5x+12}{x^2+5x+16}\right)$ internal of difference belong unity.

7) To show that

i) $\Delta\nabla = \Delta - \nabla$

ii) $\Delta + \Delta = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

8) Evaluate $\frac{\Delta^2 x^3}{Ex^3}$

9) To show that

$$\Delta^n \mathrm{Sin}(a + bx) = \left(2\mathrm{Sin}\frac{b}{2}\right)^n, \mathrm{Sin}[a + bx + \frac{n}{2}]$$

10) Evaluate

Δ Cot2^x

Interpolation with Equal internals

(Newton forward ⇒ Backward Interpolation)

Interpolation: Interpolation is the process of finding the values of y for any intermediate value of x. between a and a + nR.

Extrapolation: Extrapolation is the process of obtaining the value of y for a value of x outside the internal a and (a + nh).

Method of Interpolation

a) for equal Internal

1) Newton-Gregory forward Interpolation method

2) Newton-Backward Interpolation method

3) Sterling formula (Cen pal difference)

b) for Unequal internal

1) Lagrange's Method

2) Newton's divided difference method

Newton Care gory forward Interpolation formula for Equal internals

We know that interpolation near the beginning of the tabulated us

f (a + μh) = Eμf(a)

Using Relation E and Δ -

f (a + μh) = $(1 + \Delta)^{\mu}$ f(a)

on exporting $(1 + \Delta)^{\mu}$ by using Prenominal theorem we get.

$$f(a+\ \mu h) = \left[1 +\ \mu\Delta + \frac{\mu(\mu-1)}{L^2}\Delta^2 + \frac{\mu(\mu-1)(\mu-2)}{L^3}\Delta^3 +\ ----\right] f(a)$$

$$f(a+\mu h) = f(a) +\ \mu\Delta\, f(a) +\ \frac{\mu(\mu-1)}{L^2}\Delta^2 f(a) + \frac{\mu(\mu-1)(\mu-2)}{L^3}\Delta^3 f(a) +\ --$$

It is called as the Newton – Gregory formula for Δ interpolation

This formula can also be written as

$$y_\mu = y_0 + \mu\Delta y_0 + \frac{\mu(\mu-1)}{L^2}\Delta^2 y_0 +\ ----$$

Also written in this form

$$f(x) = f(x_0 +\ \mu h) = f(x_0) + \mu.\Delta f(x_0) + \frac{\mu(\mu-1)}{L^2}\Delta^2\ f(x_0)$$

$$+\frac{\mu(\mu-1)(\mu-2)}{L^3}\Delta^3 f(x_0) +\ ----$$

$$+ - - - + \frac{\mu(\mu-1)(\mu-2)(\mu-n-1)}{L^3} \Delta^n f(x_0)$$

where $\frac{x - x_0}{h} = \mu$

Newton Gregory formed interpolation formula is particularly useful for interpolating the value of f(x) near the beginning of the set of value given, his called interval of differencing while Δ n formed difference operator.

Example: Given below are the figures of population of a district for different years. Find the population for 1975.

Year	1951	1961	1971	1981	1991
Population (In Lakhs)	7	9	6	14	16

Solution: The difference table is give below

Year (x)	**Population f(x)**	**Δ f(x)**	**Δ^2 f(x)**	**Δ^3 f(x)**	**Δ^4 f(x)**
1951	7				
1961	9	2	–5		
1971	6	–3	11	16	–33
1981	14	8	–6	–17	
1991	16	2			

We want

f(1975) = f(a + μh) where

a = 1951, h = 10

then

a + μh + 1975

1951 + μ× 10 = 1975

10 μ = 24

$$\mu = \frac{24}{10} = 2.4$$

By Newton formula for formed interpolation we get

$$f(a+ \ \mu h) = f(a) + \mu.\Delta f(a) + \frac{\mu(\mu–1)}{L^2}\Delta^2 \ f(a)$$

$$+\frac{\mu(\mu–1)(\mu–2)}{L^3}\Delta^3 f(a) + \frac{\mu(\mu–1)(\mu–2)(\mu–3)}{L^4}\Delta^4 f(a)$$

$$f(1975) = f(1951) + 2.4.\Delta f(1951) + \frac{2.4(2.4\ –1)}{L^2}\Delta^2 f(x)$$

$$+\frac{(2.4)(2.4–1)(2.4\ –2)}{L^3}\Delta^3 f(1951)$$

$$+\frac{(2.4)(2.4–1)(2.4\ –2)(2.4\ –3)}{L^4}\Delta^4 f(1951)$$

= 7 + 4.8 – 8.4 + 3.584 + 1.1088

f(1975) = 8.0928

Hence, the population for year 1975 = 8.0928 Lakhs..

Example: from the following table of half yearly premium for polices maturing at different ages, estimate the premium for polices maturing at age of 46

Age	45	50	55	60	65
Premium (m rupees)	114.84	96.16	83.32	74.48	68.48

(UPTU-2003, 2004)

Solution: The difference table is

Age	Premium	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
45	114.84				
		-18.68			
50	96.16				
			-5.84		
		-12.84			
55	83.32				
				-1.84	
			4		
		-8.84			
					0.68
60	74.48	-6	2.84	1.16	
65	68.48				

We have,

f(46) = f(a + μh)

where, a = 45, h = 5

then

a + μh = 46

45 + μ (5) = 46

5μ = 1

μ = 1/5 = 0.2

Using Newton forward difference formula

$$f(46) = f(\mu + \mu h) = f(a) + \mu f(a) + \frac{\mu(\mu - 1)}{L^2} \Delta^2 f(a)$$

$$+\frac{\mu(\mu-1)(\mu-2)}{L^3}\Delta^3 f(a)+\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{L^4}\Delta^4 f(a)$$

$$= 114.84 + (0.2)(-18.68) + \frac{(0.2)(0.2.1)}{2}(5.84)$$

$$+\frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{24}(0.68)$$

f(46) = 110.525632

Hence the premium for polices maturing at age of 46 is = 110.52563215 Ans:

Example: find the cubic polynomials which takes the following values:

x	0	1	2	3
f(x)	1	2	1	10

Solution: The difference table is

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)
0	1			
		1		
1	2		-	
		--1		1
2	1		10	
		9		
3	1			

We have

f(x) = f(a + μh)

we have

a = 0, h = 1

x = 0 + μ– 1

μ = x

By Newton forward difference interpolation

$$f(x) = f(a) + \mu . \Delta f(a) + \frac{\mu(\mu-1)}{L^2}\Delta^2 f(a)$$

$$+ \frac{\mu(\mu-1)(\mu-2)}{L^3}\Delta^3 f(a)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$f(x) = 2x^3 - 7x^2 + 6x + 1$

Which is required cubic polynomials

Example: The following table given the marks secured by 100 students in the Numerical Analysis subjects

Range of Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	25	35	22	11	7

Use Newton's forward difference interpolation formula to find

The number of students who got were than 55 marks

The number of students who secured marks in range from 36 to 45.

Solution: The given table can be arranged as follows

Marks Obtained	No. of Students
Less than 40	25
Less than 50	60
Less than 60	82
Less than 70	93
Less than 80	100

The difference table is

Marks Obtained less than (x)	No of student (y)	Δy	Δ²y	Δ³y	Δ⁴y
40	25				
50	60	35			
60	82	22	-	2	5
70	93	11	-	7	
		7	-4		
80	100				

a = 40, h = 10 and x = 55

f(55) = f(a + μh)

55 = 40 + μ(10)

15 = 10μ

$$\mu = \frac{15}{10} = 1.5$$

Applying Newton's forward difference formula

$$f(x) = f(a + \mu h) = f(a) + \mu \Delta f(a) + \frac{\mu(\mu-1)}{L^2} \Delta^2 f(a)$$

$$+ \frac{\mu(\mu-1)(\mu-2)}{L^3} \Delta^3 f(a)$$

$$+ \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{L^4} \Delta^4 f(a)$$

$$f(55) = 25 + (1.5)(35) + \frac{(1.5)(0.5)}{2}(-13)$$

$$+ \frac{(1.5)(0.5)(-0.5)}{6}(2)$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(5)$$

= 71.6171875

f(55) $\propto$ 72

There are 72 students who got less than 55 marks

$\therefore$ No of students who got mone

than 55 marks = 100 – 72 = 28

ii) Here, we are to calculate the number of students securing marks between 36 and 45.50. So we shall calculate difference of y(45) and y(36).

To Calculate y(45)

$$\mu = \frac{x - a}{h} = \frac{45 - 40}{10} = 0.5$$

Using Newton forward difference formula

$$y(x) = y_0 + \mu \Delta y_0 + \frac{\mu(\mu - 1)}{L^2}\Delta^2 y_0 + \frac{\mu(\mu - 1)(\mu - 2)}{L^3}\Delta^3 y_0$$

$$+ \frac{\mu(\mu - 1)(\mu - 2)(\mu - 3)}{L^4}\Delta^4 y_0$$

$$y(45) = 25 + (0.5)(35) + \frac{(0.5)(-0.5)}{2}(-13)$$

$$+ \frac{(0.5)(-0.5)(-1.5)}{6}(2) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}$$

= 44.0546

y(45) $\approx$ 44

To Calculate y (36)

$$\mu = \frac{x - a}{h} = \frac{36 - 40}{10} = 0.4$$

Again using Newton forward difference formula

$$y(x) = y_0 + \mu \Delta y_0 + \frac{\mu(\mu - 1)}{L^2} \Delta^2 y_0 + \frac{\mu(\mu - 1)(\mu - 2)}{L^3} \Delta^3 y_0$$

$$+ \frac{\mu(\mu - 1)(\mu - 2)(\mu - 3)}{L^4} \Delta^4 y_0$$

$$= 25 + (-0.4)(35) + \frac{(-0.4)(-1.4)}{2}(-13)$$

$$+ \frac{(-0.4)(-1.4)(-2.4)}{6}(2)$$

$$+ \frac{(0.4)(-1.4)(-2.4)(-3.4)}{24}(5)$$

y(36) = 7.864

y(36) ≈ 8

Hence, the number of students who secured marks the range from 36 to 45 = 44 – 8 = 36

Newton Gregory Backward Interpolation formula

let the function y = f(x)

x	x_0	x_1	x_2	--------	x_n
y = f(x)	$y_0 = f(x_0)$	y1 = f(y_1)	y2 = f(x_2)	--------	$y_n = f(x)$

Let us required to evaluate f(x)

for $x = x_n + \mu h$

where x_n is last term and μ is any real number h → Internal of difference

then we have,

$y_u = f(x_n + \mu h) = E^4 f(x_n)$ {∴ E = 1 –∇}

$= (1 - \nabla)^{-4} y_n$

Exposing Binomial theorem

$$= \left[1 + 4\nabla + \frac{\mu(\mu+1)}{L^2}\Delta^2 + \frac{\mu(\mu+1)(\mu+2)}{L^3}\Delta^3\right]$$

$$y_u = f(x_n + \mu h) = y_n + \Delta y_n + \frac{\mu(\mu+1)}{L^2}\Delta^2 y_n$$

$$+ \frac{\mu(\mu+1)}{L^3}\Delta^3 y_n + ---$$

It is called Newton's backward Interpolation formula.

1) It the function y = f(x) takes the values y_0, y_1, y_2 ------ y_n corresponding the values x_0, x_0th, $x_0 + 2h$ ---- $x_0 + nh$ then Newton backward interpolation formula can be written as

$f(x) = f(x_0 + nh + \mu h) = f(x_0 + nh) + \mu . \Delta fx_0 + nh$

$$+ \frac{\mu(\mu+1)}{L^2}\Delta^2 f(x_0 + nh) + ---$$

$$+ \frac{\mu(\mu+1)(v+2)—(u+\bar{n}-1)}{L^n}\nabla^n f(x_0 + nh)$$

Where

$$\frac{x - (x_0 + nh)}{h} = u$$

2) Newton backward formula is used for finding the value of y for x_1 when x is used x_n (end).

3) It is also used for extra polating valves of y for x when x is slighty greater then x_n.

Example- Given the following data.

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

To estimate f (7.5)

Solution- The difference table is-

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$	$\Delta^7 f$
1	1							
2	8	7						
3	27	19	12					
4	64	37	18	6				
5	125	61	24	6	0			
6	216	91	30	6	0	0		
7	343	12	36	6	0	0	0	
8	51	16	42	6	0	0	0	0

To find f (7.5)

Here, x_n = last term = x

h = 1

$x = x_n + uh$

7.5 = 8 μ(1)

μ = − 0.5

By Newton Backward difference formula-

$y_u = y_{n+\mu}\ \nabla y_n + \frac{\mu(u+1)}{L2}\ \nabla^2 y_n + \frac{\mu(u+1)(u+2)}{L2}$

$\nabla^3 y_n + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{L4} \nabla^4 y_n$

$= 512 + (-0.5)\ 169 + \frac{(-0.5)(-0.5+1)}{L3}(42)$

$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{L3}(6) + 0$

= 512 – 84.5 – 5.25 – 0.375

f(7.5) = 421.875 **Ans**.

Example-Find the polynominal of degree for which takes the following values-

x	2	4	6	8	10
y	0	0	1	0	0

(UPTU-2006)

Hence or otherwise evaluate f(9)

Solution-The difference tables

The difference table

x	y = f(x)	∇ f(x)	∇^2f(x)	∇^3f(x)	∇^4f(x)
2	0				
4	0	0			
6	1	0	1	–	6
8	0	–	–	3	
10	0	0	1		

To find polynomial

$x = x_n + uh$ {where xn = Last term =10

x = 10 + u 2 h = difference = 2

$\mu = \frac{x-10}{2}$ $y_n = 0$

By Newton Backward difference formula-

$y_n = f(x) = y_n + u \nabla y_n + \frac{u(u+1)}{L2} \nabla^2 y_n$

$$+\frac{\mu(u+1)(u+2)}{L3}\nabla^3 fx+\frac{\mu(u+1)(u+2)(u+3)}{L4}\ \nabla 4y_n$$

$$f(n)=0+\left(\frac{x-10}{2}\right)(0)+\frac{\left(\frac{x-10}{2}\right)\left(\frac{x-10}{2}+1\right)}{2}(1)$$

$$+\frac{\left(\frac{x-10}{2}\right)\left(\frac{x-10}{2}+1\right)\left(\frac{x-10}{2}+2\right)}{6}(3)$$

$$+\frac{\left(\frac{x-10}{2}\right)\left(\frac{x-10}{2}+1\right)\left(\frac{x-10}{2}+2\right)\left(\frac{x-10}{2}+3\right)}{24}6$$

$$=\frac{(x-10)(x-8)}{8}+\frac{(x-10)(x-8)(x-6)}{16}+\frac{(x-10)(x-8)(x-6)}{64}$$

$$=\frac{(x-10)(x-8)}{8}\left[1+\frac{x-6}{2}+\frac{(x-6)(x-4)}{8}\right]$$

$$=\frac{(x-10)(x-8)}{8}\left[\frac{8+4x-24+x^2-10x+24}{8}\right]$$

$$=\left(\frac{x^2-18x+80}{8}\right)\left(\frac{x^2-6x+80}{8}\right)$$

$$=\frac{x^4-24x^3+196x^2 624x+640}{64}$$

$$f(x)=\frac{1}{64}x^4-\frac{3}{8}x^3+\frac{49}{16}x^2-\frac{39}{4}x+10$$

It is required polynomial

Also find

f(9) =

Example-The following table gives the polynomial of a town during the last six censuses. Estimate using any suitable interpolation formula the increase in the population during the period from 1946 to 1948.

Year	1911	1921	1931	1941	1951	1961
Population/m. thousand	12	15	20	27	39	52

Solution- The difference table is as

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1911	12					
1921	15	3				
1931	20	5	2			
1941	27	7	2	0		
1951	39	12	5	3	3	
1961	52	13	1	-4	-7	-

By Newton Backward difference formula when

$$y_u = y_n + u(\nabla y_n) + \frac{u(u+1)}{L2}\nabla^2 y_n + \frac{u(u+1)(u+2)}{L2}\nabla^3 y_n$$

$$+\frac{u(u+1)(u+2)(u+3)}{L4}\nabla^4 y_n + \frac{u(u+1)(u+2)(u+3)(u+4)}{L5}\nabla^4 y_n \_\_\_\_\_\_\_\_(1)$$

Here, x = 1946, x_n = Last term 1961,

h = 10

$x = x_n + uh$

1946 = 1961 + u (10)

u = − 1.5

Population in 1946

$$y_{1946} = 52 - 1.5\ (13) + \frac{(-1.5)(-1.5+1)}{2}(1)$$

$$+\frac{(-1.5)(-1.5+1)(-1.5+2)}{6}(-4)$$

$$+\frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)}{2.4}(-7)$$

$$+\frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{120}(-10)$$

$$= 52 = 1.5(13) + \frac{1.5(0.5)}{2} - \frac{(1.5)(0.5)(0.5)2}{3}$$

$$-\frac{(1.5)(0.5)(0.5)(1.5)}{24}\times 7 - \frac{(1.5)(0.5)(0.5)(1.5)}{12}$$

= 52 – 19.5 + 0.375 – 0.25 – 0.641 – 0.1172

= 32.3437 thousand

Population in 1946 = 32.3437 thousands

Population in 1948

Here x = 1948, x_n = 1961 h = 10

$x = x_n + uh$

1948 = 1961 + u (10) ⇒ u = − 1.3

Putting the value of x = 1948 and u = − 1.3 in Newton Backward formula (1)

$$y\ 1948 = 52 + (-1.3)\ (13) + \frac{(-1.3)(-1.3+1)}{2}$$

$$+ \frac{(-1.3)(-1.3+1)(-1.3+2)}{L3}(-4)$$

$$+ \frac{(-1.3)(-1.3+1)(-1.3+2)(-1.3+3)}{L4}(-7)$$

$$+ \frac{(-1.3)(-1.3+1)(-1.3+2)(-1.3+3)(-1.3+4)}{L5}(-10)$$

= 34.873215 thousand

Population in 1948 = 34.873215 thousand.

Increase in population from 1946 to 1948

= 34.873215 – 32.3437

= 2.52915

= 2.53 thousand (Approx)

Missing Terms

Example- Estimate the missing terms in the following.

x	1	2	3	4	5	6	7
y	2	4	8	?	32	64	128

Solution-There six value of x and y are given therefore we can assume the given function expressible as a polynomials of degree five.

i.e Δ^6 y = 0 ⇒ Δ^6 f(x) = 0

⇒ $(E-1)^6$ f(x) = 0

$$\Rightarrow \left[E^6 - 6_{C_1}E^5 + 6_{C_2}E^4 - 6_{C_3}E^3 + 6_{C_4}E^4 - 6_{C_5}E^5 + 6_{C_6}\right]f(x) = 0$$

⇒ E^6 f(x) – 6 E^5 f(x) + 15 E^4 f(x) – 20 E^3 f(x) + 15 E^2 f(x) – 6 E f(x) + f(x) = 0

⇒ f(x +6) – 6 f(x +5) + 15 f(x +4) – 20 f(x +3) + 15 f(x +2) – 6 f(x +1) + f(x) = 0

for x = 1 (First tabale value)

⇒ f(7) – 6 f(6) + 15 f(5) – 20 f(4) + 15 f(3) – 6 f(2) + f(1) = 0

⇒ 128 – 6 × 64 + 15 × 32 – 20 f(4) + 15 × 8 – 6 × 4 + 2 = 0

⇒ 128 – 384 + 480 – 20 f(4) + 120 – 24 + 2 = 0

⇒ 20 f (4) = 730 – 408

= 322

f(4) = 16.1 **Ans**.

Example- Estimate the missing terms in the following

x	0	1	2	3	4
y = f(x)	1	3	9	?	8

Solution- Four entries are given so y can be repeated by third degree polygnomial.

Δ^3 y = constant

Δ^4 y = 0

$(E-1)^4$ y = 0

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1)^4 y = 0$

$\Rightarrow E^4 y - 4E^3 y + 6E^2 y - 4Ey + y = 0$

$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y0 = 0$

$\Rightarrow 81 - 4y^3 - 6(9) - 4(3) + 1 = 0$

$\Rightarrow 81 - 4y_3 + 54 - 12 + 1 = 0$

$4y_3 = 124$

$y_3 = 31$

Hence missing terms f(3) = 31 **Ans**.

Example- Obtain the estimate of the missing figure in the following table-

x	1	2	3	4	5	6	7	8
f(x)	1	8	?	64	?	216	343	512

Solution-As six value of [x_1 f(x)] are given therefore we may. Assume the function to be represented by a polynomials of agree five so that

$\Delta^5 f(x) = \text{constant}$

$\Delta^6 f(x) = 0$

$(E - 1)^6 f(x) = 0$

$[E^6 - {}^6C_1 E^5 + {}^6C_2 E^4 - {}^6C_3 E^3 - {}^6C_2 E^2 - {}^6C_1 E + 1] f(x) = 0$

$\Rightarrow f(x + 6) - 6 f(x + 5) + 15 f(x + 4) - 20 f(x + 3) + 15 f(x + 2) - 6 f(x + 1) + f(x) = 0$ ______(1)

Put x = 1 in equation (1) we get

$\Rightarrow f(7) - 6 f(b) + 15 f(5) - 20 f(4) + 15 f(3) - f(3) - 6 + (2) + f(1) = 0$

Putting the known values, we get

$\Rightarrow 343 - 6 \times 216 + 15 f(5) - 20 \times 64 + 15 f(3) - 6 \times 8 + 1 = 0$

$\Rightarrow$ 15 f(5) + 15 f(3) = 2280

$\Rightarrow$ f(5) + f(3) = 152________________(2)

for x = 2 in equation (1) we get.

f(8) – 6f(7) + 15f(6) – 20f(5) + 15f(4) – 6f(3) + f(2) = 0

$\Rightarrow$ 512 – 6 × 343 + 15 × 216 – 20f(5) + 15 × 64 – 6 f(3) + 8 = 0

$\Rightarrow$ 20 f(5) + 6 f(3) = 2662

$\Rightarrow$ 10 f(5) + 3 f(3) = 1331________________(3)

Solving equation (2) and (3), we get

f(3) = 27 and f(5) = 125 **Ans**.

Example- Find out missing value from the following.

x	5	10	15	20	25	30
y	7	?	13	15	?	25

Solution-since here given four enteries y can be represented by the third degree polygnomal

Δ^3 f(x) = constant

$\Rightarrow \Delta^4$ f(x) = 0

$\Rightarrow (E - 1)^4$ f(x) = 0

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1)$ f (5) = 0

$\Rightarrow$ f(25) – 4 f(20) + 6 f(15) – 4 f(10) + f (5) = 0

$\Rightarrow$ f(25) – 4(15) + 6(13) – 4 f(10) + 7 = 0

$\Rightarrow$ f(25) – 4 f(10) – 60 + 78 + 7 = 0

$\Rightarrow$ f(25) – 4 f(10) = 25 ______________(1)

Again

$(E - 1)^4$ f (10) = 0

$(E^4 – 4E^3 + 6E^2 – 4E + 1) f (10) = 0$

$\Rightarrow$ f (30) – 4 f(25) + 6 f(20) – 4 f(15) + f(10) = 0

$\Rightarrow$ 25 – 4 f(25) + 6(15) – 4 (13)+ f (10) = 0

$\Rightarrow$ − 4 f(25) + f(10) = − 63 ________________(2)

Solving equation (1) and (2), we get

f(10) = 10.87

f(25) = 18.47 **Ans**.

Example- Find the missing values in the following table-

x	0	5	10	15	20	25
y	6	10	-	17	-	31

Solution- Here, there are four given entries so f(x) can be represented by third degree polynomol

Δ^3 f(x) = constant

$\Rightarrow$ Δ^4 f(x) = 0

$\Rightarrow$ $(E − 1)^4$ f(x) = 0

$\Rightarrow$ $(E^4 – 4E^3 + 6E^2 – 4E + 1)$ f (0) = 0

$\Rightarrow$ E^4 f(0) – $4E^3$ f(0) + $6E^2$ f(0) – 4 E f(10) + f (0) = 0

$\Rightarrow$ f(20) – 4 f(15) + 6 f(10) – 4 f(5) + f (0) = 0

$\Rightarrow$ f(20) – 4(17) + 6f(10) – 4 (10) + 6 = 0

$\Rightarrow$ f(20) – 68 – 6f(10) – 40 + 6 = 0

$\Rightarrow$ f(20) + 6 f(10) = 102 ______________(2)

Again from (1) use have

$\Rightarrow$ f (25) – 4 f(20) + 6 f(15) – 4 f(10) + f(5) = 0

$\Rightarrow$ 31 – 4 f(20) + 6(17) – 4 f(10)+ 10 = 0

$\Rightarrow$ 31 – 4 f(20) + 102 – 4 f(10) + 10 = 0

$\Rightarrow$ 4 f(20) + 4f(10) = 143 ________________(3)

Solving equation (2) and (3) we get

f(10) = 13.25

f(20) = 22.5 **Ans**.

Example- Find the missing values in the following table

x	45	50	55	60	65
y	3.0	-	2.0	-	− 2.4

Solution-

I[st] Method

Since only three entries-

$y_0 = 3$, $y_2 = 2$, $y_4 = -2.4$ are given the function y can be represented by a second degree polynomial

$\therefore \Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

$\Rightarrow (E-1)^3 y_0 = 0$ and $(E-1)^3 y_1 = 0$

$(E^3 - 3E^2 + 3E-1)\, y_0 = 0$ and $(E^3 - 3E^2 + 3E-1)$

$y_3 - 3y_2 + 3y_1 - y_0 = 0$

$y_4 - 3y_3 + 3y_2 - y_1 = 0$

$y_3 + 3y_1 = 0$_____________(1)

$3y_3 + y_1 = 3.6$_____________(2)

Solving (1) and (2) we get

h = 2.925	$y_2 = 0.225$

Second Method

The difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45 50 55 60	$3.0 = y_0$ y_1 $2.0 = y_2$ y_3	 $y_1 - 3$ $2 - y_1$ $y_3 - 2$	 $5 - 2y_1$ $y_1 + y_3 - 4$ $-0.4 - 2y_3$	 $3y_1 + y3 = 9$ $3.6 - y_3 - 3y_3$
65	$-2.4 = y_4$	$-2.4 - y_3$		

Since only three entries y_0, y_2, y_1 are given the function y can be represented by second degree polynomidls

$\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

3 y1 + y3 – 9 = 0, ______________(1)

$3.6 - y_1 - 3y_3 = 0$ _____________(2)

Solving 1 and 2 we get

$y_1 = 2.925$

$y_3 = 0.225$ **Ans**.

Exercise

1. Find the Lowest degree polynomidl y (x) that will fit the data. Also find y (1)

x	0	2	4	6	8
y	5	9	61	209	50

(**Ans. y = x3 – 2x + 5, y (1) = 4**)

2. Find the value of sin 30° 15′ 30″ from the following table.

Angle (x)	30°	31°	32°	33°	34°
sin x	0.5000	0.5150	0.5299	0.5446	0.5592

(**Ans. 0.5039**)

3. If lx represents the number of persons living at age x in a life table, find as accuarately as data will permit lx for valves of x = 35, 42 and 47. Given $l_{20} = 512$, $l_{30} = 439$, $l_{40} = 346$ and $l_{50} = 243$

(**Ans. $l_{35} = 354$, $l_{42} = 326$, $l_{47} = 512$**)

4. Pea press y as a polynomial in x by Newton forward difference interpolation for the following table

x	0	1	2	3	4
y	3	6	11	18	27

(**Ans.** $y = x^2 + 2x + 3$)

5. A varying current in a circuit was froms to have the values tabulated below-

lime (l sin)	0	2	4	6	8	10
Current I (amp.)	0	1.960	3.684	4.952	5.514	5.403

Estimate i when (i) t = 1 sin (ii) t = 10.5 sin

(**Ans. f(1) = 0.995, f(10.5), = 5.224**)

6. From the following table evaluable f (3.8) using Newton backward interpolation formula.

x	0	1	2	3	4
f	1.00	1.50	2.20	3.10	4.60

7. Find the cubic polynomial interpolation which takes on the value.

$f_0 = 5, f_1 = 1, f_2 = 9, f_3 = 25, f_4 = 55$

Hence find f_5

(**Ans. 1325**)

8. Find the cubic polynomial which takes the following tables-

y(0) = 1, y(1) = 0, y(2) = 1 and y(3) = 10

hence or otherwise obtain y (4)

(Ans. $x^3 - 2x^2 + 1$, 35)

9. From the following table of half yearly premium for polices maturing at different ages, estimate the premium for policy maturing at the age 63.

Age	45	50	55	60	65
Premium in types	114.84	96.16	83.32	74.48	64.48

(**Ans. 70.5851**)

11. Find the missing term in the following data

x	0	1	2	3	4
y	0	3	9	-	81

(**Ans. 31**)

12. Find interpolation the missing value in the following data.

x	0	5	10	15	20	25
y	6	10	-	17	-	31

(**Ans. 13.25, 22.5**)

13. Obtain the estimate of the missing figure in the following table-

x	2	2.1	2.2	2.3	2.4	2.5	2.6
y	0.135	-	0.111	0.100	-	0.082	0.074

(**Ans.** $y_{2.1} = 0.123$, $y_{2.4} = 0.0904$)

14. Find the missing value of the following data.

x	1	2	3	4	5
y	7	-	13	21	37

(**Ans.** $y_2 = 9.5$)

15. Assuming that the value of y belongs to a polynomidls of degree 4, complete the root three.

x	0	1	2	3	4	5	6	7
y	1	-1	1	-1	1	-	-	-

(**Ans.** $y_5 = 3$, $y_6 = 129$, $y_7 = 35$)

Chapter

Interpolation with Unequal Internals

In the previous chapter the arguments are given at equal internals. In this chapter the arguments are given at unequal internals then the various differences between two successive values are not constant. Such type interpolation evaluate can be by Newton's divided difference formula and Lagrange's formula. The formula of unequal internals can be used for equal Internals but the formulas for equal internals cannot be used in case of unequal internals.

1. Lagrange's Interpolation formula
2. Divided difference formula

Lagrange's Interpolation formula for unequal internals:

Let f(x) denotes a polynomials of nth degree which take the values $f(x_0)$, $f(x_0+h)$, $f(x_0 + 2h)$ -- + (x0 + nh) when x has values x01x1x2-------xn respectively.

The polynomials f(x) of degree n can be written as

$f(x) = A_0(x – x_1) (x – x_2)$ ------- $(x – x_n) + A_1 (x – x_0) (x – x_2)$ ------- $(x – x_n)$

+ ------- + $A_n (x – x_0) (x – x_1)$ ----- $(x – x_n–1)$ ------ (1)

where A_0, A_1 ------ An are constant to be determined.

Putting $x = x_0$, x_1 ------- x_n in equation (1) successively.

for $x = x_0$

$f(x_0) = A_0 (x_0– x_1) (x_0– x_2)$ ------- $(x_0–x_n)$

$$\therefore A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)------(x_0-x_n)}$$

Put $x = x_1$

$f(x_1) = A_1 (x_1– x_2) (x_1– x_2)$ ------ $(x_n– x_n)$

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) - - - (x_1 - x_n)}$$

1 1 1
1 1 1

for x = xn

Similarly

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_2) - - - -(x_n - x_n)}$$

Substituting the values of A_0, A_1 ------ An in equation (1) we get.

$$f(x) = \frac{(x - x_1)(x - x_2) - - - (x - x_n)}{(x_0 - x_1)(x_0 - x_2) - - - (x_0 - x_n)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2) - - - (x - x_n)}{(x_1 - x_0)(x_1 - x_2) - - - (x_1 - x_n)} f(x_1) + - - - +$$

$$+ \frac{(x - x_0)(x - x_1) - - - (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) - - - (x_n - x_{n-1})} f(x_n)$$

It is called Lagrange's Interpolation formula

Note: If f(x) denotes a polynomials of nth degree which takes the values y_0, y_1, y_2 ----- y_n when x has values x_0, x_1, x_2 ------ x_n respectively. Then Lagrange formula can be written as-

$$y(x) = \frac{(x - x_1)(x - x_2) - -(x - x_n)}{(x_0 - x_1)(x_0 - x_2) - -(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) - - - (x - x_n)}{(x_1 - x_0)(x_1 - x_2) - - - (x_1 - x_n)} y$$

$$+ - - - + \frac{(x - x_0)(x - x_1) - -(x - x_{n-1})}{(x_n - x_0)(x_n - x_1) - -(x_n - x_{n-1})} y_n$$

The Lagrange's formula can be used for both equal and unequal internals.

Example 1: The value of y and x are given as below:

X	5	6	9	11
y	12	13	14	16

find the value of y when x = 10 (UPTU- 2008, 05, 07)

Solution: let y = f(x)

We have $x_0 = 5$, $x_1 = 6$, $x_2 = 9$ and $x_3 = 11$

$f(x0) = 12$, $f(x_1) = 13$ $f(x_2) = 14$ $f(x_3) = 16$

Applying Lagrange's formula:

$$f(x) = \frac{(x-x_1)(x-x_2)---(x-x_3)}{(x_0-x_1)(x_0-x_2)---(x_0-x_n)} f(x_0)+$$

$$= \frac{(x-x_0)(x-x_2)---(x-x_3)}{(x_1-x_0)(x_1-x_2)---(x_1-x_3)} f(x)$$

$$+ \frac{(x-x_0)(x-x_1)---(x-x_3)}{(x_2-x_0)(x_2-x_1)---(x_2-x_3)} f(x_2)+$$

$$= \frac{(x-x_0)(x-x_1)---(x-x_2)}{(x_3-x_0)(x_3-x_1)---(x_3-x_3)} f(x)$$

$$\frac{(10-6)(10-9)---(10-11)}{(5-6)(5-9)(5-11)}(12)+= \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-6)(10-6)---(10-11)}{(9-5)(9-6)(9-11)}(14)+= \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(13)$$

= 14.66667

Example: The function y = f(x) is given at the point (7, 3) (8, 1) (9, 1) and (10, 9). Find the value of y for x = 9.5, using Lagrange's Interpolation formula.

Solution: from the given data, we can write

$x_0 = 7$	$x_1 = 8$	$x_2 = 9$	$x_3 = 10$
$f(x_0) = 3$	$f(x_1) = 1$	$f(x_2) = 1$	$f(x_3) = 9$

Using Lagrange's interpolation formula.

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) += \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) += \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$\frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)}(3) += \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)}(1)$$

$$+\frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)}(1) += \frac{(x-7)(x-8)(x-9)}{(10-7)(8-9)(8-10)}(1)$$

$$+\frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)}(1) += \frac{(x-7)(x-8)(x-9)}{(10-7)(8-9)(8-10)}(9)$$

Putting x = 9.5 in above

$$f(9.5) = -\frac{1}{2}(9.5-8)(9.5-9)(9.5-10) + \frac{1}{2}(9.5-7)(9.5-9)$$

$$(9.5-10) - \frac{1}{2}(9.5-7)(9.5-8)(9.5-10)$$

$$+\frac{3}{2}(9.5-7)(9.5-8)(9.5-9)$$

= 3.625 Answer

Example: By means of Lagrange's formula prove that

$y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_{-5})$ approximately

Solution: We have, Draw a table

x	–5	–3	3	5
y = y_x	y_{-5}	y_{-3}	y_3	y_5

From the above data, Lagrange's Interpolation formula is:

$$y_x = \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} += \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)} y_{-3}$$

$$+ \frac{(x+3)(x+3)(x-5)}{(3+5)(3+3)(3-5)} y_3 += \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} y_5$$

Putting x = 1 in above-

$$y_1 = \frac{(1+3)(1-3)(1-5)}{(-2)(-8)(-10)} y_{-5} += \frac{(1+5)(1-3)(1-5)}{(2)(-6)(-8)} y_{-3}$$

$$+ \frac{(1+5)(1+3)(1-5)}{(8)(6)(-2)} y_3 += \frac{(1+5)(1+3)(1-3)}{(10)(8)(2)} y_5$$

$$y_1 = -\frac{1}{5} y_{-5} + \frac{1}{2} y_{-3} + y_3 - \frac{3}{10} y_5$$

$$y_1 = -0.2 y_{-5} + 0.5 y_{-3} + y_3 - 0.3 y_5$$

Example: Apply Lagrange's formula to find f(5) give that f(1) = 2, f(2) = 4, f(3) = 8, f(4) = 16 and f(7) = 128. Explain why the result differ from 2^5.

Solution: Let $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 7$

$f(x_0) = 2$, $f(x_1) = 4$, $f(x_2) = 8$, $f(x_3) = 16$ $f(x_4) = 128$

By Lagrange's: interpolation formula. We follow

$$f_{(x)} = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} f(x_0) +$$

$$= \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} f(x_2)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} f(x_2) +$$

$$= \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} f(x_3)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} f(x_4)$$

Putting the various value and x = 5, we get

$$f(5) = \frac{(5-2)(5-3)(5-4)(5-7)}{(1-2)(1-3)(1-4)(1-7)}(2) += \frac{(5-1)(5-3)(5-4)(5-7)}{(2-1)(2-3)(2-4)(2-7)}(4)$$

$$+ \frac{(5-1)(5-2)(5-4)(5-7)}{(3-1)(3-2)(3-4)(3-7)}(8) += \frac{(5-1)(5-2)(5-3)(5-7)}{(4-1)(4-2)(4-3)(4-7)}(11)$$

$$+ \frac{(5-1)(5-2)(5-3)(5-4)}{(7-1)(7-2)(7-3)(7-4)}(128)$$

$$f(5) = {}^{-7}/_{3} + \frac{32}{5} - 24 + \frac{128}{3} + \frac{128}{3}$$

= –0.66667 + 6.4 – 24 + 42.66667 + 8.533334

= 32.93334

It is clear from the given data that the function is of the form f(2) = 2^x according to which f(5) = 25 = 32. Which differs from the interpolated value of f(5) = 32.93334

It is due to the fact that in Lagrange's formula it is being assumed that f(x) is a polynomials but here f(x) = 2^x is not polynomial.

Divided Differences

The differences defined by taking into carrideration the change in the values of the argument are called divided differences.

Let the function y = f(x) has the value $f(x_0)$, $f(x_1)$ -------- $f(x_n)$ corresponding to the values x_0, x_1, x_2 ------- x_n of the argument x. Then the first divided difference f(x) for the arguments' x_0, x_1 is defined as-

$$\frac{f(x_0)-f(x_1)}{x_0-x_1}$$

and denoted by $\underset{x_1}{\Delta}(x_0)$ or $f(x_0, x_1)$ and this symbol rodas Aitken symbol (Δ)

Thus

$$f(x_0, x_1) = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

Similarly, we can find for the arguments (x_1, x_2) (x_2, x_3) ------- (x_n, y_n)

The Second divided differences for the arguments x_0, x_1, x_2 n defined as

$$f(x_0, x_1, x_2) = \underset{x_1 x_2}{\Delta} f(x_0) = \frac{f(x_1, x_2)-f(x_0, x_1)}{x_2-x_0}$$

Similarly the third, fourth etc. divided difference can also be find.

Newton's Divided Differences formula for unequal internals

Suppose $f(x_0)$, $f(x_1)$, $f(x_2)$ ----- $f(x_n)$ be the value of f(x) corresponding to x_0, x_1, x_2 -------- x_n values of the argument x. These values of x are supposed to be unequally spaced.

The first divided difference of f(x) relative to x and x_0 is given by

$$f(x_1 x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

Which gives-

f(x) – f(x_0) = (x $-x_0$) + $(x_1 x_0)$

f(x) = f(x_0) + (x $-x_0$) + $(x_1 x_0)$ ------- (1)

Next, second divided difference of f(x), relative of x, x_0 and x_1 is given by

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

Which gives f(x, x_0) – f(x_0, x_1) = (x – x_1) f(x, x_0, x_1)

f($x_1 x_0$) = f(x_0, x_1) + (x – x_1) + (x, x_0, x_1)

$\Rightarrow \frac{f(x) - f(x_0)}{x - x_0} = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$ equation……..(1)

$\Rightarrow$ f(x) = f(x_0) + (x – x_0) f (x_0,x_1) + (x – x_0) (x – x_1) f(x, x_0, x_1) ------- (2)

Similarly,

f(x) = f(x_0) + (x – x_0) f(x_0, x_1) + (x – x_0) (x – x_1) f(x_0, x_1, x_2)

+(x – x0) (x – x1) (x – x2) f(x, x0, x1, x2) ……….. (3)

Proceeding in the same way, we get

f(x) = f(x_0) + (x – x_0) f(x_0, x1) + (x – x_0) (x – x_1) f(x_0, x_1, x_2)

+(x – x_0) (x –x_1) (x – x_2) f(x_0, x_1, x_2, x_3)

+(x – x_0) (x – x_1) (x – x_2) (x – x_3) f (x_0, x_1, x_2, x_3, x_4)

+ (x – x_0) (x – x_1) (x – x_2) (x – x_3) (x – x_4) f(x_0, x_1, x_2, x_3, x_4, x_5)

+ -----------------+----------------

+(x – x_0) (x – x_1) ------- (x – x_{n-1}) f (x_0, x_1, x_2 ------ x_n)

$+(x - x_0)(x - x_1)$ ------- $(x - x_{n-1})(x - xn)\ f(x_0, x_1, x_2$ ------ $x_n)$

The Last term of L.H.S. vanishes i.e.

$f(x, x_0, x_2$ -------- $x_n) = 0$ account of the fact that $f(x, x_0, x_1$ ------- $x_n)$ is the divided difference of f(x) of order (n + 1) and f(x) is a polynomials of degree n. Hence we have

$f(x) = f(x_0) + (x - x_0)\ f(x_0, x_1) + (x - x_0)(x - x_1)(x - x_1)\ f(x_0, x_1, x_2)$

$+ (x - x_0)(x - x_1)(x - x_2)\ f(x_0, x_1, x_2, x_3)$

$+ (x - x_0)(x - x_1)$ -------- $(x - x_{n-1})\ f(x_0, x_1, x_2$ --------- $x_n)$

This is called Newton's Divided difference formula

$f(x) = f(x_0) + (x - x_0)\ \Delta f(x_0) + (x - x_0)(x - x_1)\ \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\ \Delta^3 f(x_0)$ +------------

Example: Using the table given below find the value of (8) by Newton Divided difference formula

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Solution: The difference table is (UPTU-2008)

x	**f(x)**	**Δf(x)**	**Δ^2f(x)**	**Δ^3f(x)**	**Δf**
Δ	48				
		$\frac{100-48}{5-4} = 52$			
5	100		$\frac{97.52}{7-4} = 15$		
7	294	$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$	
			$\frac{202-97}{10-5} = 21$		
10	900	$\frac{900-294}{10-7} = 20.2$		$\frac{27-21}{11-5} = 1$	

			$\frac{310-202}{11-7} = 27$		0
11	1210	$\frac{1210-900}{11-10} 310$		$\frac{33-27}{13-7} = 1$	
			$\frac{409-310}{13-10} = 33$		0
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

Hence, $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, $x_4 = 11$, $x_5 = 13$

and $f(x_0) = 48$, $f(x_0, x_1) = \Delta f(x_0) = 52$

$f(x_0, x_1, x_2) = \Delta^2 f(x_0) = 15$

$f(x_0, x_1, x_2, x_3) = \Delta^3 f(x0) = 1$ and

$\Delta^4 f(x_0) = 0$

By Newton divided difference formula

$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0) (x - x_1) + (x - x_0) (x - x_1) (x - x_2) \Delta^3 f(x_0)$ + ------

Putting x = 8 in above formula we get.

$f(8) = 48 + (8 - 4) (52) + (8 - 4) (8 - 5) (15) + (8 - 4) (8 - 5) (8 - 7) (1) + 0$

$= 48 + 208 + 180 + 12$

= 448 Ans:

Example: Using Newton's divided formula, find polynomial function satisfying the following data

x	–4	–1	0	2	5
$f(x)_1$	1245	33	5	9	1335

Hence find f(1)

Solution: The divided difference table is as und

f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1245				
	$\frac{33-1245}{-1+4} = \boxed{-40}$			
33		$\frac{-25+404}{0+4} = 94$		
	$\frac{5-33}{0+1} = -28$		$\frac{10-54}{2+4} = \boxed{-16}$	
5		$\frac{2+25}{2+11} = 10$		$\boxed{3}$
9	$\frac{9-5}{2-0} = 2$		$\frac{88-10}{511} = 13$	
		$\frac{442-2}{5-0} = 88$		
	$\frac{1335-9}{5-2} = 442$			
1335				

Here, $x_0 = -4$, $x_1 = -1$ $x_2 = 0$ $x_3 = 2$, $x_4 = 1335$

Newton divided difference formula, we get-

$(x) = f(x_0)\ f(x - x_0)\ \Delta f(x_0) + (x - x_0)(x - x_1)\ \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\ \Delta^3 f(x_0)$

Putting the value of various difference, we find-

$(x) = 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)\ x\ (-14) + (x + 4)(x + 1) \times (x - 2)(3)$

$= 1245 - 404(x + 4) + 54(x^2 + 5x + 4) = 14(x^3 + 5x2 + 4) + 3(x^2 + 5x + 4)(x^2 - 2x)$

$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x^4 + 15x^3 + 12x^2 - 6x^3 - 30x^2 - 24x$

$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$ Ans:

Putting x = 1, we get

f(1) = 3 – 5 + 6 – 14 + 5

= –5 Ans:

Example: Prove that $\underset{bcd}{\Delta^3}\left(\frac{1}{b}\right) = \frac{-1}{abcd}$

Solution: Construct the following divided

Difference Table

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)
a	1/a			
		$\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{-1}{ba}$		
b	1/b		$(-)\frac{\frac{1}{bc}-\frac{1}{ba}}{c-a} = (4)^2\frac{1}{abc}$	
		$\frac{\frac{1}{c}-\frac{1}{b}}{c-b} = \frac{-1}{bc}$		$(+)^2\frac{\frac{1}{bcd}-\frac{1}{abc}}{d-a} = (-1)^3\frac{1}{abcd}$
c	1/c		$(-1)\frac{\frac{1}{dc}-\frac{1}{ba}}{d-b} = (-1)^2\frac{1}{bcd}$	
		$\frac{\frac{1}{d}-\frac{1}{c}}{d-c} = \frac{-1}{dc}$		
d	1/d			

from the table, we deserve that

$\underset{bcd}{\Delta^3}\frac{1}{a} = f(a,b,c,d) = \frac{-1}{abcd}$ Proved

Example: Prove that

$$\underset{yz}{\Delta^2} x^3 = x + y + z$$

Solution: Construct the following divided difference table:

x	**Entry**	**First divided difference Δ f(x)**	**Second divided difference Δ^2 f(x)**
x	x^3		
		$\frac{y^3 - x^3}{y - x} = y^3 + x^2 + yx$	$\frac{(z^2 + y^2 + zy) - (y^2 + x^2 + xy)}{z - x}$
y	y^3		$= \frac{(z^2 - x^2) + y(z - x)}{z - x}$
		$\frac{z^3 - y^3}{z - y} = z^3 + y^2 + zy$	
z	z^3		= x + y + z

from, the table

$$\underset{yz}{\Delta^2} x^2 = x + y + z$$

Example: find the polynomial of 5 degree from the following data

f(0) = –18, f(1) = 0, f(3) = 0, f(5) = –248 f(6) = 0, f(9) = 1310

Solution: Let the required polynomial of 5 degree is f(x). As f(x) = 0 at x = 1, 3 and 6, we follow

Therefore, we can write

f(x) = (x – 1) (x – 3) (x – 6) g(x) (1)

where g(x) is a polynomial of degree 2 from equation (1), we get

f(0) = (0 – 1) (0 – 3) (0 – 6) g(0) = –18 + g(0) = 1

f(5) = (5 – 1) (5 – 3) (5 – 6) g(5) = –248 ⇒ g(5) = 31

f(9) = (9 – 1) (9 – 3) (9 – 6) g(9) = –13104 ⇒ g (9) 91

for the construction of g(x), we have the following divided difference table

x	**g(x)**	**Δg(x)**	**Δ^2g(x)**
0	1		
		$\frac{31-1}{5-0} = \boxed{6}$	
5	31		$\frac{15-6}{9-0} = \boxed{1}$
		$\frac{91-31}{9-5} = 15$	
9	51		

By Newton's divided difference formula, we get,

$g(x) = g(x_0) + (x - x_0)\ \Delta\ g(x_0) + (x - x_0)\ (x - x_1)\ \Delta^2\ g(x_2)$

$= 1 + (x - 0)\ (6) + (x - 0)\ (x - 5)\ (1)$

$= 1 + 6x + x^2 - 5x$

$= 1 + x + x^2$

Therefore, we get,

$f(x) = (x - 1)\ (x - 3)\ (x - 6)\ g(x)$

$= (x - 1)\ (x - 3)\ (x - 6)\ (1 + x + x^2)$

$f(x) = x^5 - 5x^4 + 18x^3 - x^2 + 9x - 18$ Ans:

Exercise

Lagrange's Based

1. Complete f(27) from the data below:

x	14	17	31	35
f(x)	68.7	64.0	44.0	39.1

Ans: 49.3

2. Apply Lagrange's formula to find f(5) and f(6) given that

f(2) = 4, f(1) = 2, f(3) = 8, f(7) = 128

Ans; 37.8, 7, 3

3. Fit a polynomial of third degree

x	0	1	3	4
f(x)	–12	0	6	12

Ans: $x^3 - 7x^2 + 18x - 12$

4. By means of Lagrange's formula to prove that

$$y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} + y_{-1})\right]$$

5. The function y = f(x) is given at the point (7,3) (8,1) (9, 1) and (10, 9). find the value of y for x = 9.5 using Lagrange's formula

Ans: 3.625

6. Using Lagrange's method, prove that

$y_3 = 0.05\ (y_0 + y_6) - 0.3\ (y_1 + y_5) + 0.75\ (y_2 + y_1)$

7. Given that $\log_{10}300 = 2.4771$, $\log_{10}304 = 2.4D$ $\log_{10}305 = 2.4843$ and $\log_{10}307 = 2.4871$ find by using Lagrange's formula the value of $\log_{10}310$ Ans: 2.4786
8. Using Lagrange's method, prove that

$y_3 = 0.05\ (y_0 + y_6) – 0.3\ (y_1 + y_5) + 0.75\ (y_2 + y1)$

find the unique polynomial P(x) of degree 2 such that

P(1) = 1, P(3) = 27, P(4) = 6

9. Use Lagrange method of interpolation

Ans: $8x^2 – 19x + 12$

10. Given $y_0 = –12$, $y_1 = 0$, $y_3 = 6$ and $y_1 = 12$ find y_2

Ans: 4

Newton Divided Based

11. Apply Newton's divided difference formula to find the value of f(8) if f(1) = 3, f(3) = 31, f(6) = 223, f(10) – 1011 and f(11) = 1343 **Ans:** 521

12. Determine f(x) as a polynomials in x for the following data

x	–4	–1	0	2	5
F(x)	1245	33	5	9	1335

Ans: $3x^4 – 5x^3 + 6x^2 – 14x + 5$

13. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 – 2x$ Ans: 1

14. If f(0) = 8, f(1) = 11, f(4) = 68, f(15) = 123 then find the form of the function which satisfies the above values (Ans: $x^3 – x^2 + 3x + 8$)

15. Find the cubic function of x from the following data

16. Find the cubic function of x from the following data

x	0	1	2	5
f(x)	2	3	12	147

Ans: $x^3 + x^2 - x + 2$

17. If $f(x) = \frac{1}{x}$ then prove that $f(a, b) = {}^{-1}/_{ab}$

18. Find the polynomial of the lowest possible degree which takes the values 3, 12, 15, –21 when x has the values 3, 2, 1, –1 respectively **Ans:** $x^3 - 9x^2 + 17x + 6$

19. Find the form of the function $y = \mu_x$ given that $\mu_0 = 8$, $\mu_1 = 11$, $\mu_4 = 68$ $\mu_5 = 123$
Ans: $x^3 - x^2 + 3x + 8$

Fit a polynomials and fixed f(1), f(8)

x	–1	0	3	6	7
f(x)	3	–6	39	8.22	1611

Ans: x4 – 3x3 + 5x2 – 6

20. Given,
$\log_{10}654 = 2.8156$
$\log_{10}658 = 2.8182$, $\log_{10}659 = 2.8189$, $\log_{10}661 = 2.82$

find the value of log10656 by the divided difference formula

Ans: 2.81679996

Module – IV

Test of Hypotheris and Anova

Sampling Theory –

A hypothesis is an assumption about the population parameter to be tested based on sample information. An aggregate of objects under study is called population or universe. A universe containing a finite number of individuals or members is called a finite inverse.

A finite subset of a universe is called Sample. A sample is thus a small portion of the universe. The process of selecting a sample form a population is called sampling.

A large volume of statistical information is collected by way of periodic enumeration known as censuses. A census involves a complete count of every individual numbers of the population of interest, such as persons in state, students in a college, shops in a city and so on. A part from the coast and the large amount of resource that are required, the main problem is the time required to process data. Hence the results are known after a big gap of time.

Hypothesis Testing –

The term hypothesis derives from the Greek, hypotitheni derives from the Greek, hypotitheni meaning 'to put under 'or 'to suppose'. A Tentative conjecture explained on observation, phenomenon or scientific problem that can be texted by further observation, investigation and for experimentation. It refers the process of selecting and using a sample statistic to draw an inference about a population parameter based from the population.

A statistical hypothesis is a claim about an unknown population parameter value. The statistical constants of the population such as mean, the variance etc. are known as the parameter. The methodology that eroble a decision maker to draw inferences about population characteristics by analyzing the difference the corresponding hypotherized parameter value is called Hypothesis testing.

Meaning and Definition

A hypothesis is a statement about the population parameter. Hypothesis testing/significance testing is a procedure that helps us to decide whether the hypotherized population parameter value is to be accepted or rejected by making use of the information obtained from the sample.

According to the Prof. Morris Hamburg:-

"A hypothesis in statistics is simply a quantitative statement about a population."

According to Palmer O Johnson:-

"A hypothesis as island in the uncharted seas of thought to be used as bases for consolidation and recuperation as we advance into the unknown.

Characteristics of Hypothesis:-

Hypothesis must possess the following characteristics:-

1) Hypothesis should be clear and precise. If the hypothesis is not clear and precise. The inference down on its basis cannot be taken as reliable.
2) Hypothesis state relationship between variable if it happens to be a relational hypothesis.
3) Hypothesis should be limited in scope and must be specific. A researcher must remember that narrower hypothesis are generally more testable and he should develop such hypothesis.
4) Hypothesis should be consistent with most known facts i.e. it must be consistent with a substantial body of established facts. Mother words, it should be one which judges accept or being the most likely.
5) Hypothesis should be amenable to testing within a reasonable time. One should not use even an excellent hypothesis. If the same cannot be tested in reasonable time for one cannot spend a life time collecting data to test it.

Sources of Hypothesis

Hypothesis may be developed from various sources. Some of the important sources are the following:-

1) **A hypothesis arises from intuition –**

These hypothesis have no clear connection with the large body of social science. The intuition is associated with an individual, who is influenced by environment.

2) **A hypothesis also arises from other studies –**

The finding of a study may be formulated as hypothesis. The hypothesis followed in one study previously can be used in the present study.

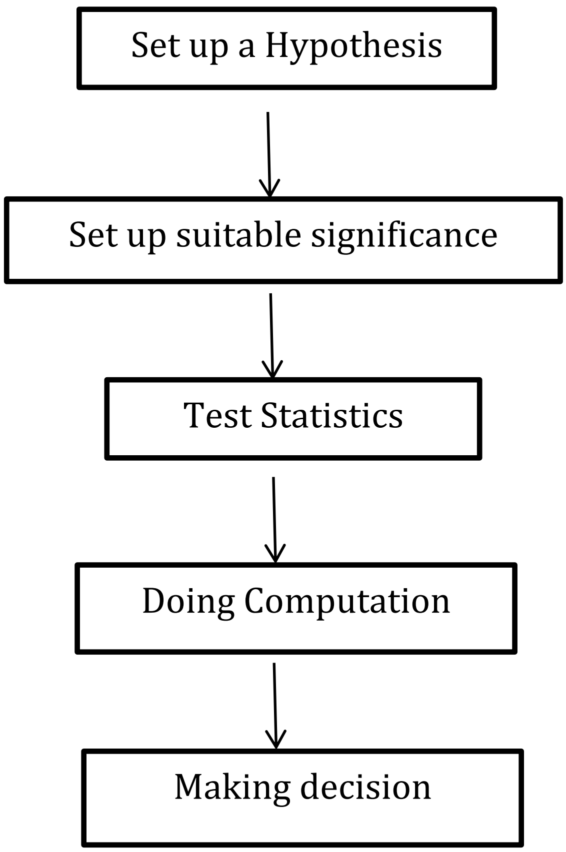
3) **Theory is a Fertile Seed-Boy of Hypothesis –**

Individuals who are rising in status are individuals who are rising in status are likely to be favorable inclined, towards individuals and objects that are helping their unlift in life. The researcher may identify the variables which influence the status of the said individuals.

4) **Personal Happiness Provide Scope for Hypothesis –**

The phenomenon of personal happiness has been studies in great detail. Happiness has been correlated with income, education, occupation, social class and so on.

Procedure of hypothesis testing



Step (1) Set up a Hypothesis –

The First thing in Hypothesis testing is to setup a hypothesis about a population parameter. Then we collect sample data produce sample statistic, and use this information to decide how likely it is that our hypothesized population parameter is cannot.

The two hypothesis in a statistical test are normally referred to as –

(i) Null Hypothesis (H_0)

(ii) Alternative Hypothesis (H_1)

(i) Null Hypothesis (H_0) – A statistical hypothesis which is stated for the purpose of possible acceptance is called null hypothesis. It is usually devoted by the symbol (H_0).

According to Prof. R.A. Fisher- "Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true".

The null hypothesis is always expressed in the form of an equation making a claim regarding the specific value of the population parameter. That is-

$$H_0 : \mu = \mu_0$$

Where μ is population mean and μ_0 represent hypothesized parameter value.

(ii) Alternative Hypothesis (H_1) – Any hypothesis which is complementary to the null hypothesis is called an alternative hypothesis is usually denoted by H_1. That is, an alternative hypothesis must be true when the null hypothesis is found to be false. In other words, the alternative hypothesis states that specific population parameter value is not equal to te value stated in the null hypothesis and is

Written as - $H_1 : \mu \neq \mu_0$

Consequently- $H_1 : \mu < \mu_0$

Or $H_1 : \mu > \mu_0$

Step 2) Set up a suitable significance level – Having set up the hypothesis, the next step is to test the validity of H_0, against that of H_1 at a certain level of significance. The hypothesis are tested on a pre-determined level of significance and as such the some should be specified. Generally in practice, either 5% level or 1% level is adopted for the purpose. The factor That affect the level of significance are –

a) The magnitude of the difference between sample means.

b) The size of the samples.

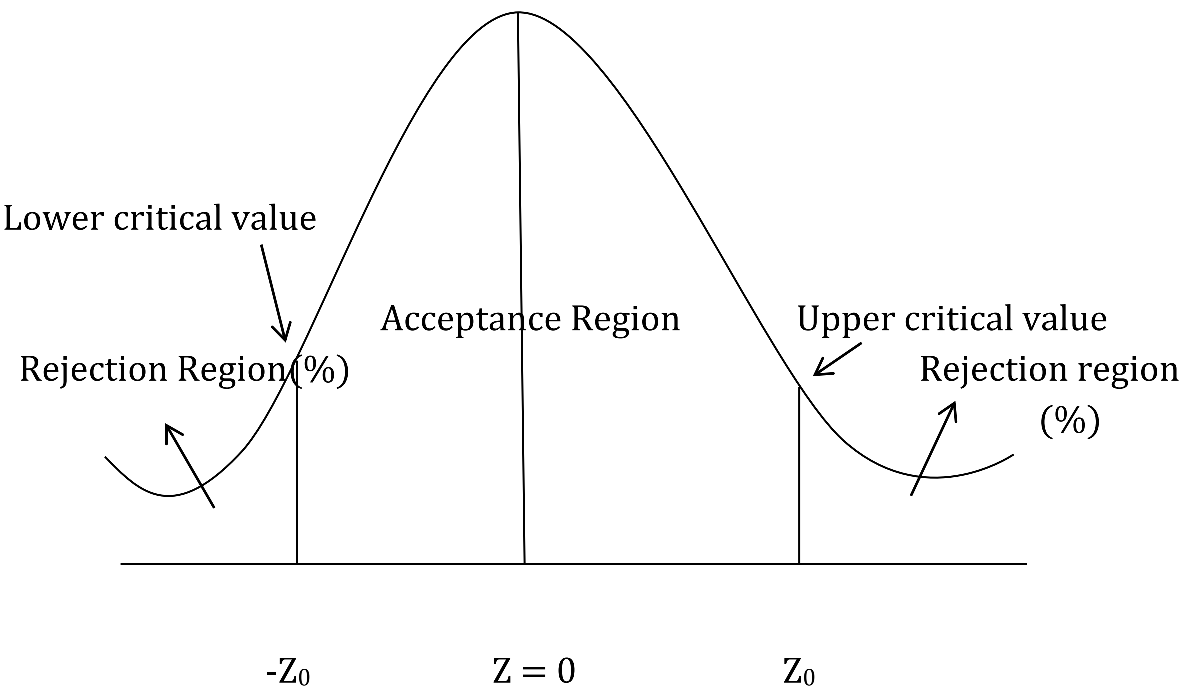
c) The variability of measurements within sample and

d) Whether the hypothesis is directional or non-directional.

In brief, the level of significance (α) must be adequate in the context of the purpose and nature of enquiry.

The probability of a statistic belonging to the critical region is known as the level of significance.

Actual Decision	Decision	
	Accept H_0	Reject H_0
H_0 is True	Correction decision Probability = 1 - α	Wrong decision type I error probability = α
H_0 is False	Wrong decision type II error probability = β	Correct decision probability = 1 - β



Two tacted Test (Level of significance α)

Step 3) Test statistic (or test criterion)- The statistic used to test a hypothesis is called a test statistic.

The next step is to compute an appropriate test statistic which is based on an appropriate probability distribution. It is used to test whether the null hypothesis set-up should be accepted or rejected. For this purpose, we use Z distribution under normal curve for large, sample ashore the sample size is equal to or larger than 30 ($n \geq 30$) and the distribution for small sample where the sample size n, less than 30 ($n < 30$).

We compute the test statistic Z under the null hypothesis for the larger sample corresponding to the statistic t, the variable $Z = \frac{t-E(d)}{S.E(t)}$ is normally distributed with in sum 0 are variance 1.

The value of Z given above under the null hypothesis is known as test statistics.

Step 4) Doing computations- Having taken the first three steps, we have completely designed a statistical test. We now proceed to the fourth step performance of various computations – form a random sample of size n, necessary for the test. These calculations include the testing statistics and the standard error of the testing statistic.

Step 5) Making Decision – Lastly, a decision should be arrived as to whether the null hypothesis is to be accepted or rejected. In this regard the value of text statistic computed to test the hypothesis plays a very important role .

Compare the calculated value of test statistic with the critical value (also called standard table of the test statistic). The decision rules for Null hypothesis are as follows –

- $|\text{Test statistic}|_{cal} = |Z|_{cal} \geq |Z|_{table}$; Reject H_o
- $|\text{Test statistic}|_{cal} = |Z|_{cal} \leq |Z|_{Table}$; Accept $H_{o=}$

Mother words, if the calculated absolute value of a test statistic is more absolute value of a test statistic is more than or equal to its critical (on table) value, then reject the null hypothesis, otherwise, accept it.

Errors in Hypothesis testing (Type I and Type II errors)

Two types of errors are generally committed in hypothesis testing. They are type I error and Type II error.

Type I Error – It is committed when we reject a correct or true hypothesis. Type I error (of rejecting a null hypothesis when it is true) is denoted by α

Thus, α = probability of type I error

= Probability of rejecting H_0 when H_0 is false.

Type II Error – it is committed when we accept a wrong or incorrect hypothesis Type II error (of accepting a null hypothesis when it is not true) is denoted by β.

Thus, β = Probability of Type II Error

= Probability of accepting H_o when H_o is not true.

If the difference between two means is zero and if our test indicated rejection of the null hypothesis we commit Type I Error.

If on the other hand the difference between two means is not zero but air test suggests acceptance of null hypothesis, we commit Type II Error.

The following table gives an idea about the Type I and Type II Errors.

	Condition	
Decision	H_o : True	H_0 : False
Accept H_o Reject H_o	Correct Decision Type I Error (α)	Type II Error (β) Correct Decision

One – Tailed and Two – Tailed tests

Basically, there are three kinds of problems of test of hypothesis, they include –

i) Two – trailed tests

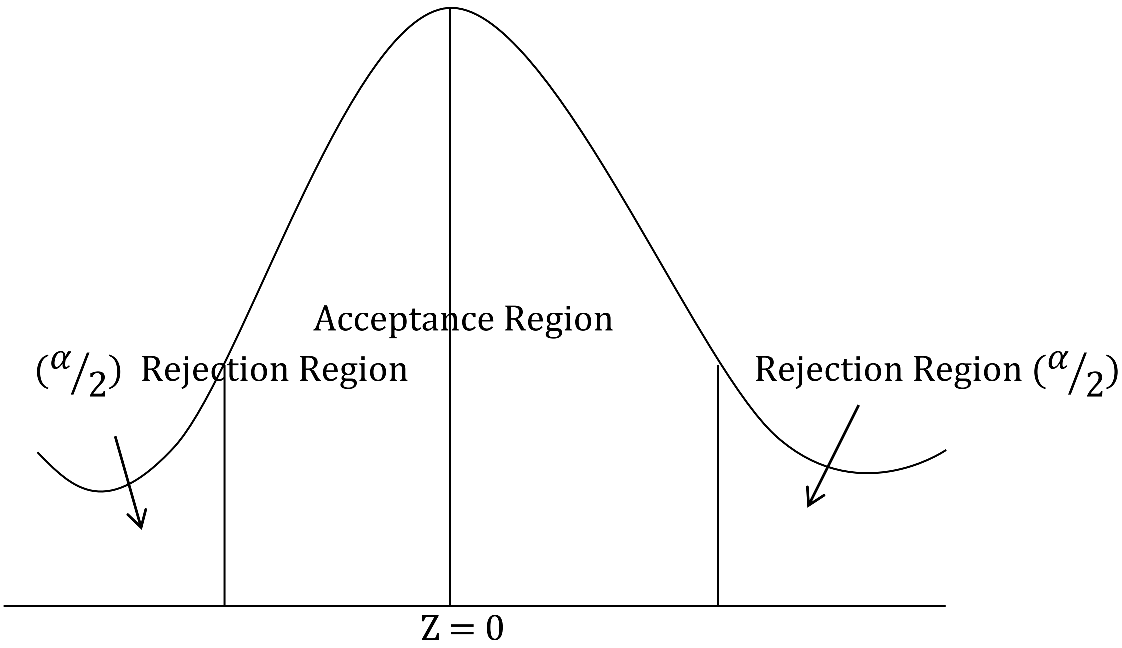
ii) Right - trialed test and

iii) Left – trailed Test

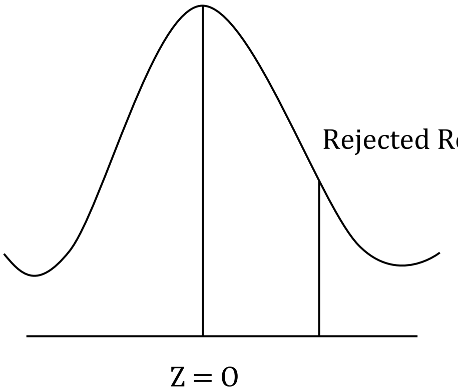
A One – sided test is a statistical hypothesis test in which the values for which we can reject the null hypothesis H_o are located entirely in one tail of the probability distribution.

A Two – sided test is a statistical hypothesis test in which the values for which we can reject the null hypothesis H_o are located in both tails of the probability distribution.

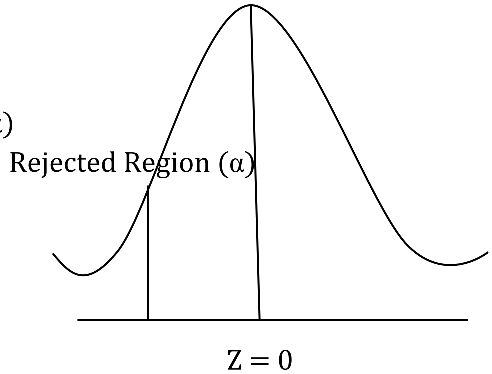
The critical region (or the region of rejection) which is generally 5% is kept on both sides of the normal distribution in the two tailed test. It means that 2.5% of the critical region is on the extreme left of the normal curve and 2.5% of the extreme right. The middle 95% is the acceptance region. In a single tail test the 5% area would be either on the extreme left of the normal curve or on the extreme right. The remaining 95% area would be the acceptance region.



Two – tailed test (α)



Right - tailed test



Left - tailed test

Procedure for testing of Hypothesis –

Following are the main steps in a testing a statistical hypothesis-

i) **Null hypothesis:-** Set up the null hypothesis H_o

ii) **Alternative Hypothesis:-** Set up the alternative hypothesis H_1

iii) **Level of Significance (α)-** Choose the level of significance.

iv) **Test Statistic -** Compute the test statistic

$$Z = \frac{t - E(t)}{SE(t)}$$

Under the null hypothesis H_o

v) **Conclusion -** Compare the computed value Z in step 4 with the critical value on significant value Z_∞ at given significance level α.

If,

$|Z|_{cal.} < |Z|_{\alpha table} \longrightarrow H_o$ Accepted

$|Z|_{cal.} > |Z|_{\alpha table} \longrightarrow H_o$ Rejected

Test of SIGNIFICANCE

Various tools of significance can be broadly classified into two categories.

i) Large sample Test

ii) Small sample Tests or Exact tests.

Large sample tests –

In this section we will discuss the tests of significance when sample are large i.e. when the samples are of the size (n>30). We have seen that for large value of n, the number of tails, almost all the distribution e.g. binomial, poissom etc are very closely approximated by normal distribution.

We shall now discuss the following for large sample test-

i) Testing the significance of population proportion.

ii) Testing the significance of the difference in proportion.

iii) Testing the significance of population mean.

iv) Testing the significance of the difference between two means.

i) Testing the significance of population proportion

Test of hypothesis involving proportions of the same attribute are useful in many situations. For example, a production firm may be concerned about the proportion of defective in small sample tests (or Exact tests). If the sample size is less than 30 i.e. (n<30) then those simples may be regarded as small samples. As a rule the methods and the theory of small sample are applicable to large samples. But the method and theory of large samples are not applicable to small samples. The small samples are used in testing a given hypothesis to find out the observed values, which could have arisen by sampling fluctuations from some values given in abvance.

For small samples, exact sampling distribution of the test statistics have been obtained. In this section, we will study the properties of t, γ^2 and F-statistics and tests of significance based on these statistic.

Student t-distribution (t-Test)

The greatest contribution to the theory of small samples was made by sir William Gossett and R. A. Fishar Gossett published his discovery in 1905 under the pen name ‘students’ and it is popularly known as t-test or students t-distribution or students distribution.

When the sample size is 30 or less and the population standard deviation is unknown we can use t-distribution

$$\mathrm{T} = \frac{\overline{x} - \mu}{s/\sqrt{n}}$$

Where

$$\mathrm{S} = \sqrt{\frac{\sum(x-x)2}{n-1}}$$

$\overline{\mathrm{x}}$ is mean of sample, μ is population mean. S is the standard deviation of population and n is sample size.

If the standard deviation of the sample ‘s’ is given then t-statistic is defined as –

$$\mathrm{t} = \frac{x - \mu}{\frac{s}{\sqrt{n-1}}}$$

The relation between s and S is

$$ns^2 = (n-1)S^2$$

Degree of freedom (d.f.)

Degree of freedom is the number of independent observations used in the making of the statistics. If k is the number of independent constant in a set of data of n observation then the degree of freedom will be (n-k). It is denoted by v^- .

$$\textbf{d.f. } (v^-) = n-k$$

Tests based on t-distribution

For small samples (n<30), the following test of significance are based on t-distribution.

i) Test for an assumed mean is the test for

$$H_0 : \mu = \mu_0$$

ii) Testing the significance of the difference between two means i.e. the test $H_0 : \mu_1 = \mu_2$ when,

1) The two samples are independent.
2) The two samples are dependent (paired t-distribution).

Application of the t-distribution

The following are some important applications of the t-distribution:-

i) Test the significance of the Mean of the random sample.
ii) Test for difference of mean of two small samples.
iii) Test of hypothesis about the difference between two means with dependent samples.
iv) Test of hypothesis about coefficient of correlation.

Test 1 – To test the significance of the mean of a random sample –
In determining whether the mean of the sample drawn from normal population deviates significantly from a stated valued (the hypothetical values of the population mean) when variance of the population is unknown we calculate the statistic

$$t = \frac{x - \mu}{\frac{s}{\sqrt{n}}}$$

Where $\overline{x}$ = mean of sample

μ = actual or hypothetical means of the population

n = the sample size

s = the standard deviation of the sample.

d.f. = v^- = n – 1

$t_{cal} < t_{Table}$ ⟶ H_o Accepted

$t_{cal} > t_{Table}$ ⟶ H_o Rejected

Example O :- A machine is designed to produce insulting washers for electrical devices of average thickness of 0.25cm. A random sample of 10 washers was fund to have an average thickness of 0.24cm with a standard deviation of 0.002cm. The test significance of the deviation. Value of t for 9 d.f. at 5% is 2.262.

Solution- Null hypothesis H_0 : μ = 0.025 i.e. the difference between $\overline{x}$ and μ is not significant or $\overline{x} = \mu$ (Two tailed test)

Alternative Hypothesis $H_1 = \mu \neq 0.025$

Given, $\overline{x}$ = 0.024, μ = 0.025, s = 0.002, n = 10

Test statistic (value of t) is

$$t = \frac{x - \mu}{\frac{s}{\sqrt{n-1}}} = \frac{0.024 - 0.025}{\frac{0.002}{\sqrt{10-1}}} = \frac{-0.001}{0.0067} = -0.149$$

d.f. v^- = n – 1 = 10 – 1 = 9

for d.f. at 5% level of significance critical value of t(given) = 2.262

$t_{cal}(-0.149) < t_{Table}(2.262)$ ⟶ H_o Accepted

- Difference is not significant

Example O:- The mean life time of sample of 100 fluorescent light bulbs produce by a company is complete to be 1570 hours with a standard deviation of 120 hours. The company claim that the average life of the bulbs produced by it is 1600 hours.

Using the level of significance of 5% is the claim acceptable.

Solution- Given,

$\overline{X}$ = 1570, s = 120, n = 100, μ = 1600

$$t = \frac{x - \mu}{\frac{s}{\sqrt{n}}}$$

$$\frac{1570-1600}{\frac{120}{\sqrt{100}}} = \frac{1570-1600}{12} = -2.5$$

At 5%(0.05) the level of significance t = 1.96

t_{cal} (2.5) > t_{Table} (1.96) $\longrightarrow$ H_o Rejected

- Claim is to be rejected.

Example O:- Ten individuals are chosen at random from a population and their heights are found to be in inches 63, 63, 64, 65, 66, 69,69,70,70,71. Discuss the suggestions that the mean height of universe is 65. (for 9 d.f. at 5% level of significance t = 2.262)

Solution:

x	d = x – 67	$d^2 = (x - 67)^2$
63	-4	16
63	-4	16
64	-3	9
65	-2	4
66	-1	1
69	+2	4
69	+2	4
70	+3	9
70	+3	9
71	+4	16
$\sum x = 670$		$\sum(x-\overline{x})^2 = 88$

Mean, $\overline{x} = \frac{\sum x}{n} = \frac{670}{10} = 67$

S.D. $s = \sqrt{\frac{\sum(x-x)2}{n-1}} = \sqrt{\frac{88}{9}} = 3.13$

then t- statistic

$$t = \frac{x-\mu}{\frac{s}{\sqrt{n}}}$$

$$= \frac{67-65}{\frac{3.13}{\sqrt{10}}} = \frac{2\sqrt{10}}{3.13}$$

t calculated $= 2.02$

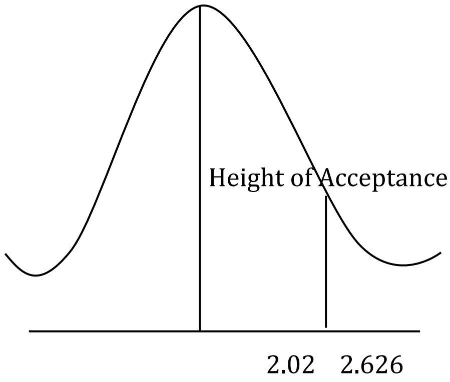
Hence, d.f. = 10 – 1 = 9

Given at 5% def. with 9d.f. table value of t = 2.262

Now,

$t_{cal}(2.02) < t_{Table}(2.262) \longrightarrow H_o$ Accepted

- Mean height of universe is 65 inches.



Example:- A random sample of size 16 value from a normal population showed a mean of 53 and a sum of square of deviation from the mean equal to 150. Can this sample be regarded as taken from the population having 56 as mean.

Obtain 95% and 97% confidence limits of the mean of the population

$V = 15, \alpha = 0.05, t = 2.131$

$$t = \frac{x - \mu}{\frac{s}{\sqrt{n}}}$$

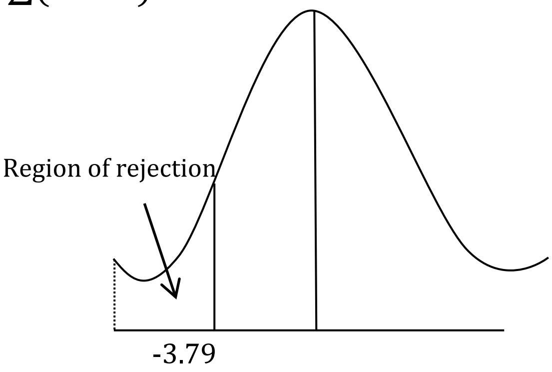
Solution: given,

$$\mu = 56,\ n = 16,\ \overline{x} = 53 \text{ and } \sum(\boldsymbol{x} - \boldsymbol{x})^2 = 150$$

then S.D. $s = \sqrt{\frac{\sum(x-x)2}{n-1}} = \sqrt{\frac{150}{15}} = \sqrt{10}$

$s = \sqrt{10}$

then t-statistic,



$$t = \frac{x - \mu}{\frac{s}{\sqrt{n}}} = \frac{53 - 56}{\frac{\sqrt{10}}{\sqrt{16}}} = \frac{-3 \; x \; 4}{\sqrt{10}}$$

$t = -3.79$

$|t| = 3.79$

When, $\alpha = 0.05$ then $t_{cal} > t_{table}$

$\alpha = 0.01$ then $t_{cal} > t_{table}$ H_0 Rejection

Thus the sample cannot be regarded as taken from the population.

Example O:- A sample of 20 items has mean 42 units and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with 45 units.

Solution:

Null hypothesis H_0: there is no significant difference between the sample mean and the population mean i.e. $\mu = 45$ units

Alternative hypothesis $H_1 : \mu \neq 45$ (two tailed led)

Given

$N = 20, \bar{x} = 42, s = 5, v = 19$d.f.

(t-test) statistic under H_0

$$t = \frac{x - \mu}{\frac{s}{\sqrt{n-1}}} = \frac{42 - 45}{\frac{5}{\sqrt{19}}} = -2.615$$

$$|t| = 2.615$$

The tabulated value of t at 5% level for 19d.f. is $t_{0.05} = 2.09$

Hence $t_{cal}(2.615) > t_{table}(2.09) \longrightarrow H_0$ Rejected

- There is significant difference between the sample mean and population mean.
- i.e. the sample could not have come from tis population

Test 2: t-test for difference of means of two small sample-

Let two independent random sample of size n_1 and n_2 with means $\bar{x}_1$ and $\bar{x}_2$ and standard deviation s_1 and s_2 we may be interested interesting the hypothesis that the sample come from the same normal population. To carry out test, we calculate the statistics as follows-

$$t = \frac{\bar{x}1 - \bar{x}2}{s\sqrt{\frac{1}{n1} + \frac{1}{n2}}} = \frac{x1 - x2}{s} \text{ x } \sqrt{\frac{n1n2}{n1+n2}}$$

Where,

$\bar{x}_1$ = mean of the first sample

$\bar{x}_2$ = mean of the second sample

n_1 = number of observation in the first sample

n_2 = number of observation in the Second sample

s = combined standard deviation.

Degree of freedom = $n_1 + n_2 - 2$

Note 1. If the two sample's S.D. s_1 and s_2 are given then we have

$$S^2 = \frac{(n1s1+n2s2)2}{n1+n2-2}$$

Note 2. If s_1, s_2 are not given then

$$S^2 = \frac{\sum(x1-\overline{x1})2+\sum(x2-\overline{x2})2}{n1+n2-2}$$

Example:- Two independent sample of 8 and 7 items respectively had the following values of the variable (weight in ances)

Sample 1	9	11	13	11	15	9	12	14
Sample 2	10	12	10	14	9	8	10	

Is the difference between the means of the sample significant.
(Given for $v^{-} = 13$ $t_{5\%} = 2.16$)

Solution: Assumed mean x_1 =12

Assumed mean x_2 =10

x_1	$(x_1 - 12)$	$(x_1 - 12)^2$	x_2	$(x_2 - 10)$	$(x_2 - 10)^2$
9	-3	9	10	0	0
11	-1	1	12	2	4
13	1	1	10	0	0
11	-1	1	14	4	16
15	3	9	9	-1	1
9	-3	9	8	-2	4
12	0	0	10	0	0
14	2	4	-	-	-
94	**-2**	**34**	**73**	**3**	**25**

Hence $\overline{x_1} = \frac{\sum x1}{n} = \frac{94}{8} = 11.75$

$$fx_1^2 = \frac{\sum(x1-12)2}{n} - (\frac{\sum(x1-12)}{n})$$

$$= \frac{34}{8} - \frac{(-2)2}{8} = 4.1875$$

And

$$\overline{x_2} = \frac{\sum x2}{n} = \frac{73}{7} = 10.43$$

Then

$$Fx_2{}^2 = \frac{\sum(x2-10)2}{n} - \left(\frac{\sum(x2-10)2}{n}\right)2$$

$$= \frac{25}{7} - \left(\frac{3}{7}\right)2 = 3.388$$

Then

$$s = \sqrt{\frac{\sum(x1-\overline{x1})2+\sum x2-\overline{x2})2}{n1+n2-2}}$$

$$= \sqrt{\frac{34+25}{8+7-2}} = \sqrt{\frac{59}{13}} = \sqrt{4.54}$$

$$= 2.13$$

Then t-statistic

$$t = \frac{\overline{x1}-\overline{x2}}{s\sqrt{\frac{1}{n1}+\frac{1}{n2}}} = \frac{11.75-10.43}{2.13\sqrt{\frac{1}{8}+\frac{1}{7}}} = \frac{1.32}{1.103}$$

$t_{calculated} = 1.2$

degree of freedom = $n_1 + n_2 - 2$

$= 8 + 7 - 2 = 13$

Table value at 5% level of significance for 13 d.f. = 2.16

$t_{cal}(1.2) < t_{table}(2.16) \longrightarrow H_0$ Accepted

- Difference between the means of sample is not significant.

Example O:- Samples of sizes 10 and 14 were taken from two normal populations with S.D. 3.5 and 5.2 The sample means were found to be 20.3 and 18.6 Test weather the means of the two population are the same at 5% level.

Solution: given

$\overline{x_1} = 20.3$, $\overline{x_2} = 18.6$, $n_1 = 10$, $n_2 = 14$, $s_1 = 3.5$,
$s_2 = 5.2$

Then

$S^2 = \frac{(n1s1+n2s2)2}{n1+n2-2} = 22.775$

$S = 4.772$

t-statistic

$$t = \frac{x1-x2}{s\sqrt{\frac{1}{n1}+\frac{1}{n2}}} = \frac{20.3 - 18.6}{4.772\sqrt{\frac{1}{10}+\frac{1}{14}}}$$

t = 0.8604

degree of freedom = $n_1 + n_2 - 2$

= 10 + 19 – 2 = 22

Table value of t at 5% level of significance for 22d.f. is t = 2.0739

Hence $t_{calcuted}$ (0.8604) < t_{table} (2.0739) ⟶ H_0 Accepted

- There is no significant difference between their means.

Exercise O

1) A random sample of size 16 has 53 as mean. The sum of square of te deviation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean. Obtain 95% and 99% confidence limits of the means of the population.

(Ans |t| =4 , H_0 Rejected)

2) A sample of 6 persons in an office revealed an average daily working of 10,12,8,9,16.5 cigarettes. The average level of smoking in the whole office has to be estimated at 90% level of confidence.
(The value of t at 5% for 5d.f. – 2.015)

(Ans 6.12, 13.08)

3) A fertilizer mixing machine is set to give 12kg of nitrate for quintal bag of fertilizer. Ten 100kg bags are examined. The percentage of nitrate per bag are as follows: 11, 14, 13, 12, 13, 14, 11, 12

If there any reason to believe that the machine is defective. (Value of t for 9d.f. is 2.262)

(Ans |t| = 1.59, H_0 reason to believe that machine is defective)

4) A certain stimulus administered to each of 12 patients reacted in the following increase in the blood pressure 5, 2, 8, -1, -3, 0, 6, -2, 1, 5, 0, 4 can it

be calculated that stimulus is accompanied by an increase in blood pressure (given that for 11d.f. at 5% level of significance is 2.201)

(Ans $|t| = 2.3701$ stimulus B.P. will income)

5) The height of 6 randomly chosen sailor in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.

(Ans $t = 0.1569$, H_0 significant)

6) Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results.

Horse A	28	30	32	33	33	29	39
Horse B	9	30	30	24	27	29	

Test whether you discriminate between two horses.

(Ans Yes, with 75% confidence)

F-test (or Snedecor's Variance Ratio Test) (Small Sample)

The f-test was first developed by the statistician R.A. fisher. The test is also known as Fisher's F-test or simply F-test is based on F-distribution.

The object of the F-test is to discover whether the two independent estimates of population variance differ significantly or whether the two sample mag be regarded as drawn from the normal population having the same variance. For confying out the test of significance. We calculate the ratio F

$$F = \frac{s_1^2}{s_2^2}$$

Where, variance $s_1^2 = \frac{\sum(x1-\overline{x1})2}{n1-1}$ and variance $s_2^2 = \frac{\sum(x2-\overline{x2})2}{n2-1}$ n_1 and n_2 refer to the number of observation in sample I and sample II respectively. It should be noted that s_1^2 is always the larger estimate of variance.

i.e. $s_1^2 > s_2^2$

or

$$F = \frac{Larger\ estimate\ of\ variance}{Smaller\ estimate\ of\ varience}$$

$v_1 = n_1 - 1$ and $v_2 = n_2 - 1$

v^{-}_1 = degree of freedom for sample having large variance

v^{-}_2 = degree of freedom for sample having smaller variance

The calculated value of F is compared with the table value for v_1 and v_2 at 5% or 10% level of significance.

$$F_{cal} > F_{table} \longrightarrow H_0 \text{ Reject}$$

$$F_{cal} < F_{table} \longrightarrow H_0 \text{ Accept}$$

Since F test is based on the ratio of two variance, it is also known as the variance Ratio test.

Assumption of F-test

The theoretical assumption on which F-test is based one

1) The populations for each sample must be normally distributed with identical mean and variance.
2) All same observation must be randomly selected and independent.
3) The ratio of s_1^2 and s_2^2 should be equal to or greater than 1.
4) Since the F-distribution is always formed by a ratio of squared values, it can never be a negative number.
5)The total variance of the various sources of variance should be additive.

Total sum of square = sum of square between the group + Sum of squares within the group.

Application of F-distribution

F-test is used to test

1) Whether two independent samples have been drawn from the normal population with the same variance.
2) Whether the two independent estimates of the population variance are homogenous or not.

Example O:- Two sources of raw material are under consideration of a company both sources seem to have similar characteristics, but the company is not sure about their respective uniformity. A sample of ten lots from source A yields a variance of 225 and a sample of eleven lots from source B yields a variance of 200. Its likely that the variance of source A is significantly greater than the variance of source B, $\alpha = 0.01$

Solution:

Here given source A : $n_1 = 10$, $s_1^2 = 225$

source B : $n_2 = 11$, $s_2^2 = 200$

Now estimated population variance in source it

$$S_1^2 = \frac{n1}{n1-1} x\ s_1^2$$
$$= \frac{10}{10-1} x\ 225$$
$$= 250$$

Estimated population variance in source B

$$S_2^2 = \frac{n2}{n2-1}$$
$$= \frac{11}{11-1} x200$$
$$= 220$$

Null Hypothesis – The variance of source A is not significantly different from the variance of source B.

F-test, $F = \frac{S_1^2}{S_2^2}$

$$= \frac{250}{220} = 1.13$$

Degree of freedom $v^-_1 = n_1 - 1 = 10 - 1 = 9$

$$v^-_2 = n_2 - 1 = 11 - 1 = 10$$

The table value of F for $v_1 = 9$ and $v_2 = 10$ at 10% level of significance = 4.913

Hence,

$$F_{cal}(1.13) < F_{table}(4.93) \longrightarrow H_0 \text{ Accepted}$$

It may be concluded that the variance of source A is not significantly greater that the variance of source B at $\alpha = 0.01$

Example O:- In a sample of 8 observations, the sum of squared deviations of items from mean was 94.5 in another sample of 10 observations. The value was found to be 101.7. Test whether the difference is significant at 5% level.

(given at 5% level of significant critical value of F for $v_1 = 7$ and $v_2 = 9$ =3.20)

Solution:

Let us take the hypothesis that the difference in the variance of the two samples is not significant.

Null hypothesis, $H_0 : s_1^2 = s_2^2$

Alternative hypothesis $H_1 : s_1^2 \neq s_2^2$

Given

$n_1 = 8 \qquad \Sigma(x_1 - \bar{x}_1)^2 = 94.5$

$n_2 = 10 \qquad \Sigma(x_2 - \bar{x}_2)^2 = 101.7$

Calculation of s_1^2 and s_2^2

$$s_1^2 = \frac{\Sigma(x1 - \overline{x1})2}{n1-1} = \frac{94.5}{7} = 13.5$$

$$s_2^2 = \frac{\Sigma(x2 - \overline{x2})2}{n2-1} = \frac{101.7}{9} = 11.3$$

F-test statistic

$$F = \frac{s_1^2}{s_2^2} = \frac{13.5}{11.3} = 1.195$$

given F critical value at $v_1 = 7$, $v_2 = 9$ with 5% level of significance = 3.29 then

$$F_{cal}(1.19) < F_{critical}(3.29) \longrightarrow H_0 \text{ Accepted}$$

And conclude that the difference in the variance of two samples is not significant at 5% level.

Example O:- Two independent sample of sample of size 7 and 6 had the following values.

Sample A	28	30	32	33	31	29	34
Sample B	29	30	30	24	27	28	

Examine whether the sample have been drawn from normal population having the same variance.

Solution: The sample have been drawn from normal population with same variance under H_0,

F test statistic $F = \frac{s_1^2}{s_2^2} \quad (s_1^2 > s_2^2)$

Computation of s_1^2 and s_2^2

x_1	$x_1 - \bar{x}$	$(x_1 - \bar{x})^2$	x_2	$x_2 - \bar{x}_2$	$(x_2 - \bar{x}_2)^2$
28	-3	9	29	1	1
30	-1	1			
32	1	1	30	2	4
33	2	4	30	2	4
31	0	0			
29	-2	4	24	-4	16
34	3	9	27	-1	1

			28	0	0
	28				26

Hence, $\bar{x}_1 = 31$, $n_1 = 7$, $\sum(x_1 - \bar{x}_1)^2 = 28$

$\bar{x}_2 = 28$, $n_2 = 6$, $\sum(x_2 - \bar{x}_2)^2 = 26$

Now

$$S_1^2 = \frac{\sum(x_1 - \bar{x}_1)^2}{n_1 - 1} = \frac{28}{6} = 4.666$$

and

$$S_2^2 = \frac{\sum(x_2 - \bar{x}_2)^2}{n_2 - 1} = \frac{26}{5} = 5.2$$

Then

F – statistic

$$F = \frac{S_1^2}{S_2^2} = \frac{5.2}{4.666} = 1.1158$$

d.f. $v_1 = n_1 - 1$ $\quad$ $v_2 = n_2 - 1$

$= 6 - 1$ $\quad$ $= 7 - 1$

$= 5$ $\quad$ $= 6$

Critical value of F at 5% level of significance for $v_1 = 5$ and $v_2 = 6 = 4.39$

Thus,

$F_{cal}(1.1158) < F_{critical}(4.39) \longrightarrow H_o$ Accepted

- There is no significant difference between the variance i.e. sample have been drawn from the normal population with same varience.

Example O – The two random reveal the following data.

Sample No.	Size	Mean	Variance
I	16	440	40
II	25	460	42

Test whether the sample come from the same normal populations.

<u>**Solution**</u> - A normal population has two parameters namely the mean μ and the variance σ^2. To test whether the two independent sample have been drawn from the same normal population , then to test

(i) The equality of means

(ii) The equality of variance

Since the t-test assumes that the sample variance are equal. We first apply F-test.

<u>F-test</u>- H_o : $\sigma_1^2 = \sigma_2^2$

The population variance do not differ significantly.

Now the Null Hypothesis H_o

The F-test statistic $F = \frac{S_1^2}{S_2^2}$ $(s_1^2 - s_2^2)$

Given, $n_1 = 16$ $n_2 = 25$

$S_1^2 = 40$ $S_2^2 = 42$

Then $F = \frac{S_1^2}{S_2^2} = \frac{\frac{n_1 S_1^2}{n_1 - 1}}{\frac{n_2 S_2^2}{n_2 - 1}} = \frac{16 \; x \; 40}{15} \text{ x } \frac{24}{25 \; x \; 42}$

$= 0.975^2$

The table value of F at d.f. $v_1 = n_1 - 1 = 16 - 1 = 15$, and $v_2 = n_2 - 1 = 25 - 1 = 24$ for 5% level of significance = 2.11

$F_{cal}\,(0.9752) < F_{table}(2.11) \longrightarrow H_o$ Accepted

→i.e. population variance are equal.

t-test

$H_o : \mu_1 = \mu_2$ i.e. population means are equal.

$H_1 : \mu_1 \neq \mu_2$

Given, $n_1 = 16$ $n_2 = 25$

$\bar{X}_1 = 440$ $\bar{X}_2 = 460$

$S^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2} = \frac{16x40 + 25X42}{16+25-2}$

$S = \sqrt{43.333}$

$S = 6.582$

t - statistic

$t = \frac{\overline{X_1} - \overline{X_2}}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{440 - 460}{6.582\sqrt{\frac{1}{16} + \frac{1}{25}}}$

$= -9.490$

And d.f. $= n_1 + n_2 - 2$

$= 16 + 25 - 2 = 39$

For 5% level of significance = 1.96

Hence,

$|t|_{cal}(9.490) > t_{Table}(1.96) \longrightarrow H_o$ is Rejected

→ there is significance difference between means i.e. $\mu_1 \neq \mu_2$

Since there is significant difference between means and no significant difference between variance, we conclude that the sample do not come from the same normal population.

Exercise O

1) Given the following information about drawn from two nrmal population $n_1 = 8$, $\sum(x_1 - \bar{x}_1)^2 = 94.5$, $n_2 = 10$ and $\sum(y - \bar{y})^2 = 101.7$
Test the equality of two population variance.
(Ans: F = 1.195, Accepted)

2) The following figure relate to the number of limits produced per shift by two workers A and B for a number of days –

A	19	22	24	27	24	18	20	19	25		
B	26	37	40	35	30	30	40	26	30	35	45

Can it be inferred that A is more stable worker compared to B ? Answer using F-test at 5% level of significance.
(Ans, F = 3.8, rejected)

3) The daily wages in Rupees of skilled workers in two cities are as follows-

	Size of sample of workers	S.D. of wages in the sample
City A	16	25
City B	13	32

(Ans – Accepted)

4) From the following two sample values, find out whether they have come from the same population.

Sample A	17	27	18	25	27	29	27	23	17
Sample B	16	16	20	16	20	17	15	21	

(Ans, Rejected)

5) In a Laboratory experiment two sample gave the following results.

Sample	Size	Sample Mean	Sum of square of deviation from the mean
1	10	15	90
2	12	14	108

Test the equality of sample variance at 5% level of significance.
(Ans- F = 1.018, Accepted, population have same variance.)

Chi-square test (Υ^2-test)

(For small sample)

Introduction

In Z-test, t-test and F-test we make assumption about the population values or parameter, therefore these tests are called parameter, therefore these tests are called parametric test. Since these test do not required any information regarding population distribution (binomial, poison or normal) these test are also called distribution-free testing method.

Here we will discuss, the chi-square (Υ^2-test) which belongs to this category of method to test a hypothesis.

The symbol Υ, is the greek letters 'chi'. The sampling distribution of Υ^2 is called Υ^2- distribution like other hypothesis testing procedure , the calculation value of Υ^2-test statistic is compared with its critical (or table) value to know whether the null hypothesis is true.

The decision of accepting a null hypothesis is based on how 'close' the sample results are to the expected results.

Definition

The chi-square, test in one of the simplest and most commonly used non-parametric tests in statistical work. The Greek letter Υ^2 is used to devote this test. The quality Υ^2 describes the magnitude of discrepancy between the observed and expected frequencies. The value of Υ^2 is calculated as –

$$\Upsilon^2 = \sum_{n=1}^{n}\left[\frac{(O_{1-} E_1)^2}{E_1}\right]$$

Where O_i → observed frequencies

E_i → expected or theoretical frequencies

Condition For CHI-square test

There are some condition which are necessary for chi-square test

i) The number of cells should be independent

ii) The sample under study must be large and total of cell frequency should not be less than 50.

iii) The cell frequency of each cell should be greater than 5. If any cell has frequency less than 5, ten it should be combine with the next or proceding cell unit the total frequency exceeds 5.

iv) The cell frequencies should not involves any logarithmic, exponential or trigonometric relation.

Properties of Chi-square distribution

i) The distribution of Υ^2 lie in the first quadrant.
ii) The range of Υ^2 distribution is from 0 to ∞.
iii) The value of Υ^2 is always positive.
iv) The value of Υ^2 will be zero if each pair is zero.
v) Υ^2 distribution has only one parameter n which is its degree of freedom (d.f.).
vi) The mean and variance of the Υ^2 distribution with n d.f. are

$$\text{mean} = n, \quad \text{variance} = 2n$$

vii) It has a unimodal curve.

Use of Chi-square (Υ^2)-

i) Test of independence.
ii) Test of goodness of fit.
iii) To test if the hypothetical value of the population variate is σ^2.
iv) To test as a test of homogeneity.
We shall mainly use the first two test.

1) **Chi-square test of goodness of fit**

The Karl Pearson developed a test for testing the significance of the discrepancy between observed (experimental) values and the theoretical values obtained under some theory or hypothesis. This test is known as Υ^2-test of goodness of fit and is used to test if the deviation between experiment (observed) values and the theory may be attributed to chance.

$$\Upsilon^2_{cal} \leq \Upsilon^2_{table} \rightarrow H_o \text{ Accept uniformally}$$

$$\Upsilon^2_{cal} > \Upsilon^2_{table} \rightarrow H_o \text{ Reject not uniformally}$$

Degree of freedom (d.f.)

Case 1 – if the data is given in the form of a series of variables in arrow or column then the degree of freedom = (no. of items in the series) – 1

$$= n - 1$$

Case 2 – When the no. of frequencies are put in cell in a contingency table.
The degree of freedom = (R – 1) (C - 1)
Where R is number of row,
C is number of column.

Working rule to calculate Υ^2

Step 1- Calculate the expected frequencies (E_i)
Step 2- Calculate the difference between each observed O_i are the corresponding expected frequency E_i for each class i.e. to find ($O_i - E_i$)
Step 3- Square the difference obtain in step 2 for each value .i.e. calculate $(O_i - E_i)^2$.
Step 4- Divide $((O_i - E_i)^2$ by the expected frequency E to get $\frac{(Oi - Ei)2}{E_i}$
Step 5- Add all these obtained in step 4
Then

$$\Upsilon^2 = \sum_{n-1}^{n} \frac{(Oi - Ei)2}{E_i}$$

Example O – A die is thrown 90 times and the number of faces shown are as indicated below-

Faces	1	2	3	4	5	6
Frequency	18	14	13	15	14	16

Test whether the die is fair.
Solution – We set up the hypothesis the die is fair, thus with this hypothesis the expected frequency for each face will be $E_i = \frac{90}{6} = 15$

$$\Upsilon^2 = \sum_{n-1}^{n} \frac{(Oi - Ei)2}{E_i} = \frac{(18-15)^2}{15} + \frac{(14-15)^2}{15} + \frac{(13-15)^2}{15} + \frac{(15-15)^2}{15} + \frac{(14-15)^2}{15} + \frac{(16-15)^2}{15}$$

$$= \frac{1}{15}[\, 9 + 1 + 4 + 0 + 1 + 1\,] = \frac{16}{15} = 1.07$$

Degree of freedom = n – 1 = 6 – 1 = 5
From table Υ^2 at 5 d.f. for 5% level of signature = 11.07

$\Upsilon^2_{cal}(1.07) < \Upsilon^2_{table}(11.07) \rightarrow H_o$ Accepted
→Conclude that the die is fair

Example 2 – 100 Students of a management institute obtained grades in statistics paper

Grade	A	B	C	D	E	Total
Frequency	15	17	30	22	16	100

Using Υ^2 test, examine the hypothsis that distribution of grade is uniform.

Solution – Here we setup null hypothesis that the grades are uniformly distributed among the students. According to H_o the expected frequency of grades will be

$$E_i = \frac{Total\ number\ of\ students}{n}$$

$$= \frac{100}{5} = 20$$

The calculation of Υ^2 is shown in the table

S. no.	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(Oi - Ei)2 / E_i$
1	15	20	-5	25	1.25
2	17	20	-3	9	0.45
3	30	20	10	100	5.00
4	22	20	2	4	0.20
5	16	20	-4	16	0.80
					$\Sigma = 7.7$

Hence, $\Upsilon^2 = \Sigma \left(\frac{(Oi - Ei)2}{E_i}\right) = 7.7$

Degree of freedom = n – 1 = 5 -1 = 4

Table value, at 5% level of significance = 9.49

$$\Upsilon^2_{cal}(7.7) < \Upsilon^2_{table}(9.49) \rightarrow H_o \text{ Accept}$$

Conclude that distribution of grades is uniform.

2. **Υ^2-test for Independence of Attributes** (In a Contingency Table)

The Chi-square test can be used to find out whether two or more attributes are associated or not. This test helps in finding the association or independence of two or more attributes. In order to test the independence of two attributes are independent. In case of contingency table the expected frequency will be calculated as –

$$\text{Expected frequency } F_i = \frac{Row\ Total\ x\ Column\ Total}{Grand\ Total}$$

Degree of freedom = (R – 1) (C – 1)

R → number of row

C → number of column

Note: 1) Acceptance of H_0 means that the two attributes are independent and there is no association between them.

2) Rejection of H_o leads to the conclusion that H_1 is true and data support the hypothesis that there is some relationship between the two variables.

Example O:- The following table gives a classification of a sample of 160 plants of their flower colour are flatness of leaf.

	Flat Leaves	Coloured Leaves	Total
White Flower	99	36	135
Red Flower	20	5	25
Total	119	41	160

Test whether the flower colours is independent of the flatness of leaves.

Solution –

Null Hypothesis H_0 : There is no association between colour of flowers and flatness of leaves.

Alternative Hypothesis H_1 : The flower colour depends upon the flatness of leaves.

Expected frequency $= \frac{Row\ Total\ x\ Column\ total}{Grand\ Total}$

	Flat Leaves	Coloured Leaves	Total
White Flower	$\frac{135\ x\ 119}{160} = 100.41$	$\frac{135\ x\ 41}{160} = 34.59$	135
Red Flower	$\frac{25\ x\ 119}{160} = 18.59$	$\frac{25\ x\ 41}{160} = 6.41$	25
Total	119	41	160

Calculated of Υ^2

Observed Value (O_i)	Expected Value (E_i)	$O_i - E_i$	$(O_i - E_i)^2$	$(Oi - Ei)2/_{E_i}$
99	100.41	-1.41	1.99	0.019
20	18.59	1.41	1.99	0.107
36	34.59	1.41	1.99	0.056
5	6.41	-1.41	1.99	0.310
			Total	0.492

Degree of Freedom = (R-1)(C-1)

= (2-1)(2-1) = 1

The critical (table) values of Υ^2 at $\alpha = 0.05$

For 1.d.f. = 3.841

Decision

$\Upsilon^2_{cal}(0.492) < \Upsilon^2_{table}(3.841)$

→ H_0 Accepted

→H_0 not rejected

They are independent fitness of leaves with colour of flower.

There is no association between them.

<u>Example O</u> – The theory predicts the proportion of beans in the fair group G_1, G_2, G_3 and G_4 should be in the ratio 9:3:3:1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental results support the theory ? (UPTU-2008)

Solution –

Null Hypothesis :H_0: The experimental result support the theory i.e. there is no significant difference between the observed and theoretical frequency.

Under H_0, the theoretical (expected) frequency can be calculated as follows-

$E(G_1) =$